Is bitcoin the new digital gold?
Evidence from extreme price movements in financial markets

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This version

January 4, 2020

Abstract

Is bitcoin the new digital gold? To answer this question, we investigate the potential benefits of bitcoin during extremely volatile periods. We use the multivariate extreme value theory, which is the appropriate statistical approach to model the tail dependence structure of the return distribution. First, considering positions in equity markets, we find - similarly to previous studies - that the correlation of extreme returns increases during stock market crashes and decreases during stock market booms. Second, by combining each equity market with bitcoin, we find that the correlation of extreme returns sharply decreases during both market booms and crashes, indicating that bitcoin could provide the sought-after benefits of diversification during turbulent times. A similar result is obtained for gold, confirming its well-recognized status as a safe haven when a crisis occurs. Finally, we find a low extreme correlation between bitcoin and gold, which implies that both assets can be used together in times of turbulence in financial markets to protect equity positions. From a portfolio management perspective, we show that the introduction of bitcoin (along with gold) substantially improves the performance of equity positions under tail risk constraints. Such evidence indicates that bitcoin can be considered the new digital gold. However, gold itself can still play an important role in portfolio risk management.

Keywords: bitcoin; diversification benefits; extreme correlation; extreme value theory; gold; portfolio risk management; tail dependence

JEL classification: C46; F38; G01

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[*] We would like to thank Menachem Brenner, Stephen Chan, Michel Crouhy, Saralees Nadarajah, the participants at the *Mathematics for Industry: Blockchain and Cryptocurrencies* conference at the University of Manchester (Manchester, September 2018) and at the *International Risk Management* conference at Bocconi University (IRMC, Milan, June 2019) for their comments. The usual disclaimer applies.

1. Introduction

Extreme adverse events in financial markets always represent a painful experience for market participants. Thus, portfolio diversification during extremely volatile periods is of utmost importance for asset managers, financial advisors and investors to control the risk levels of their portfolios. A shift to secure assets during such periods is a strategy that is very frequently used to reduce portfolio riskiness. Over time, gold has played the role of a safe haven (see Ranaldo and Söderlind, 2010); the yellow metal has been considered to be a suitable flight-to-quality investment choice for portfolio diversification and portfolio hedging against adverse price movements (see Jaffe, 1989; Hillier et al., 2006; Baur and Lucey, 2010; Baur and McDermott, 2010, among others). Investors are accustomed to including gold in their portfolios, as it provides thoughtful diversification benefits to traditional asset classes and is characterized by a high level of liquidity. Moreover, the purchasing power and the value of gold have remained stable under the threat of the erosion of the monetary or banking systems. Gold has over 5,000 years of history as a safe haven.

Over the past few years, bitcoin has made an explosive entrance into the financial world. Bitcoin is an online communication protocol that uses a virtual currency, which allows electronic payments. Ten years after the seminal paper by Nakamoto (2008) introducing bitcoin, the cryptocurrency has been a success in terms of popularity among both individual and institutional investors. Essentially, bitcoin has been introduced as something "new" with possible revolutionary effects on the traditional financial system. Following Böhme et al. (2015), bitcoin and other cryptocurrencies represent a "social science laboratory" with potential disruptive innovations based on blockchain technology (payment services, money transfers, and transaction settlements in the banking and financial sectors, for example). Although it is no longer the only cryptocurrency, bitcoin is by far the largest in terms of market capitalization. Furthermore, the usefulness of bitcoin has sparked the interest of both academics and practitioners in the areas of statistics, risk management and asset management. Specifically, the financial community has been wondering whether bitcoin could provide diversification benefits in times of crises by looking for a rigorous answer to the following question: "Is bitcoin the new digital gold?".

In this paper, we investigate the potential diversification benefits of bitcoin for equity positions in periods of extreme price volatility. To this end, we use the extreme value theory, which is the appropriate approach to study this issue and has proven its practical value in various disciplines, including earth sciences, engineering, environmental sciences and more recently finance (see e.g. de Haan and Ferreira, 2006). In a multivariate framework, we focus on extreme correlation, which summarizes the tail dependence structure of the return distribution. We develop a research strategy in *four* steps. First, we consider as a starting point a position in equity markets (Europe and the United States) and find that the extreme correlation increases during stock market crashes and decreases during stock market booms. This confirms a well-known stylized fact regarding the international equity markets. Second, we combine each equity market with bitcoin and find that the correlation of extreme returns decreases sharply during both market booms and crashes, indicating that bitcoin can play an important role in portfolio risk management in times of market crash. Third, we combine each equity market with gold and find a similar result, thus confirming the well-recognized status of gold as a safe haven. Fourth, we study the joint behavior of bitcoin and gold and find a low extreme correlation, indicating that both assets can be useful together in times of turbulence in financial markets. These results point to the explicit importance of incorporating assets with low levels of extreme correlation in a portfolio. In fact, our portfolio analysis does suggest that including bitcoin (along with gold) in equity positions substantially improves the portfolio performance under tail risk constraints. Such evidence indicates that bitcoin can be considered to be the new digital gold. However, gold itself can still play an important role in portfolio risk management.

This paper is organized as follows. Section 2 details the research strategy followed in this study. Section 3 deals with the modeling of extremes. Section 4 introduces the estimation process, presents the testable hypotheses and reports the empirical results. Section 5 discusses the general economic backdrop of bitcoin and gold, compares the potential diversification benefits of bitcoin and gold in a separate manner, and then assesses the joint potential of bitcoin and gold as diversifiers. Section 6 provides several robustness checks. Section 7 assesses the practical implications of the extreme dependence structure for asset management. Section 8 concludes the paper putting forward that bitcoin is indeed the new digital gold.

2. Research strategy

This section presents our research strategy to investigate the potential diversification benefits of bitcoin in asset management during extremely volatile periods.

Our objective is to answer the following question: "Is bitcoin the new digital gold?". In light of this, we focus on extremely volatile periods, since such market conditions are a primary concern for investors intending to protect their equity positions from large losses. We use the multivariate extreme value theory, which is considered to be a rigorous statistical tool to model the dependence structure of the return distribution far away from the center (see Tawn, 1990; Coles and Tawn, 1991, among others). Interestingly, it also provides an attractive alternative to the assumption of an underlying multivariate normal distribution, which is usually violated when modeling asset returns (see e.g. Richardson and Smith, 1993). We focus on extreme correlation which is a widely used measure in the context of multivariate extreme value analysis for summarizing the tail dependence structure. From a practical standpoint, correlation measures stand for key indicators to assess the degree of diversification for portfolio decisions. Our research strategy unfolds in four steps, which are described below in detail.

Step 1: Equity markets

We consider a position in equity markets (Europe and the United States). We focus on the correlation in equity markets when extreme price movements happen so as to assess potential diversification benefits in equity positions. Several empirical studies have found that the correlation of extreme returns increases during stock market crashes and decreases during stock market booms (Longin and Solnik, 2001; Ang and Bekaert, 2002; Ang and Chen, 2002; Hartmann et al., 2004; Goetzmann et al., 2005). Indeed, correlation is not related to market volatility per se, but to the market trend. This implies that the probability of large losses in the two markets is significantly higher than the probability of large gains, since downside market conditions constitute the driving force in equity correlation. A possible explanation for this behavior can be found in Campbell and Hentschel (1992). They suggested that news arrival can affect volatility, which, therefore, results in an increase in the risk premium and a decrease in asset prices. The decrease is greater in case of bad news, whereas it is less sharp in case of good news. In that connection, Poon et al. (2003) noted that bad news are more likely to influence equity markets, generating at the same time a co-movement in their volatility and a stronger dependence during turbulence periods.

The objective of this first step is to confirm the stylized fact about equity markets that the correlation of extreme returns increases during stock market crashes and decreases during stock market booms. This lack of international diversification when it is most needed is a negative consideration of integration in equity markets, starting with the advent of globalization when international investing became more important (see Levy and Sarnat, 1970; Black and Litterman, 1992; Solnik, 1995; You and Daigler, 2010, among others). The recent global financial crisis of 2008 that triggered the worst economic contraction since the 1930s, reminded investors of the harmful consequences of contagion in international equity markets in a painful way (Aloui et al., 2011; Bekaert et al., 2014; Derrien and Kecskés, 2013). In such market conditions, investors would have to consider alternative diversification strategies to detect potential benefits.

Step 2: Equity markets and bitcoin

We then combine each equity market with bitcoin. The objective of this second step is to assess the potential diversification benefits of bitcoin in times of extreme price fluctuations. In accordance with Caballero and Krishnamurthy (2008), when investors have similar portfolios (e.g. equity portfolios), the inclusion of a different type of assets (e.g. bitcoin) allows investors to diversify their portfolio better in times of financial turmoil. The usefulness of bitcoin for investors is characterized by a decreasing extreme correlation during market crashes, implying a desirable alternative with respect to diversification benefits. In contrast, an increasing extreme correlation during market downturns would imply limited diversification benefits by including bitcoin in an equity position.

Step 3: Equity markets and gold

We then combine each equity market with gold. Several empirical studies have found a low correlation between equity markets and gold during financial crises. According to Baur and McDermott (2010), gold can be viewed as a “panic buy in the immediate”, following an extreme negative shock which takes place in equity markets; in other words, gold loves bad news according to a Wall Street adage. The objective of this third step is to confirm the well-known status of gold as a safe haven during stock market crashes. By examining the extreme correlation between equity markets and gold, we expect to find a decreasing extreme correlation, thus confirming its well-recognized status of a safe haven when a crisis occurs.

Step 4: Bitcoin and gold

Finally, further motivated by the idea that bitcoin could provide attractive properties as a diversifier, we consider a position in bitcoin and gold. The objective of this fourth step is to see whether both assets can provide diversification benefits together in periods of stress. The usefulness of including both bitcoin and gold in an equity position would be characterized by a decreasing extreme correlation during market crashes as complementary diversifiers, thus implying extra diversification benefits. In contrast, an increasingly extreme correlation during market crashes would imply limited diversification benefits, as bitcoin and gold could be viewed as substitutable assets.

3. Modeling approach

This section describes the modeling approach for the behavior of extreme returns in financial markets. We model the bivariate tail dependence structure of the distribution of asset returns. We define extreme returns as return exceedances, that is, returns lower than a threshold for the left tail (negative return exceedances) and returns higher than a threshold for the right tail (positive return exceedances). First, we deal with the univariate modeling of extremes by fitting a general Pareto distribution (GPD) for each marginal distribution of return exceedances using the peaks-over-threshold method. Second, we deal with the multivariate modeling of extremes by fitting the Gumbel-Hougaard copula and focusing on the extreme correlation defined as the correlation of return exceedances. Third, we provide some statistical tests related to normality and dependency based on the extreme correlation estimates to gain a better statistical understanding of the dependence structure as the events become more extreme.

3.1 Univariate modeling of extremes

The univariate extreme value theory aims to quantify the stochastic behavior of a random variable at its unusually high (or low) levels. In particular, univariate extreme value analysis refers to the estimated probability of events occurring that are more extreme than those that have already been observed. Such rare events can be described via statistical distribution functions.

Consider a sequence of independent and identically distributed random variables $X = \{X_1, X_2, \dots, X_n\}$ with a continuous cumulative distribution function F_X. For positive extremes, over threshold $u > 0$, the distribution of exceedances $(X - u)$, denoted by F_X^u, is given by the following:

$$F_X^u(x) = Pr(X - u \leq x | X > u) = \frac{F_X(u + x) - F_X(u)}{1 - F_X(u)}, 0 < x \leq x_{F_X} - u \qquad (1)$$

where $x > 0$ represents a given value of exceedances and $x_{F_X} \leq +\infty$ is the right endpoint of F_X. The peaks-over-threshold method is an efficient method for modeling the extremes over a specific threshold under an unknown distribution (see Leadbetter, 1991). In particular, with this method, an appropriate threshold is first selected in order to exclude the observations not belonging in the right tail (observations below the threshold u). Then, a specific distribution is fitted by using only those values that exceed this threshold (observations over the threshold u). The main advantages of this method are that: (i) it links asymptotic models when the threshold converges towards the endpoint of the distribution (see below) and (ii) leads to a likelihood function that provides a simple way to integrate the non-stationarity of the threshold excesses. The peaks-over-threshold method is based on the theoretical arguments of Pickands (1971) who showed that when the number of exceedances is a non-homogeneous Poisson process, the distribution of the number of exceedances follows a Poisson distribution, while their random size follows a GPD.[2] Accordingly, this representation (often referred to as a Poisson-GPD model) considers two distributions: one for the number of exceedances in a specific time period and one for their size. From a practical point of view, we are interested only in the size of exceedances in our modeling approach (see Section 4), as the return variable is the key input in asset management and portfolio risk assessment (see Section 7).

One of the main attractions resulting from Pickands (1971) is that for a large class of underlying distributions, Balkema and de Haan (1974), and Pickands (1975) showed that the excess distribution F_X^u can be asymptotically approximated as $u \to +\infty$ by a GPD (a result often referred to as Pickands–Balkema–de Haan theorem). The GPD is denoted by $G_{\xi,\sigma}$, given by the following:

$$G_{\xi,\sigma}(x) = 1 - p\left(1 + \frac{\xi(x-u)}{\sigma}\right)^{-1/\xi}, x > u \tag{2}$$

where $x > 0$ represents a given value of exceedances, p represents the tail probability of exceedances over a sufficiently high threshold u $\left(p = 1 - G_{\xi,\sigma}(u)\right)$, $\sigma > 0$ is the scale parameter and $\xi \in \mathbb{R}$ is the tail index. This theoretical result about the limit distribution of exceedances holds when the threshold u goes to the upper endpoint x_{F_X} of the distribution. In practice, as the dataset contains a finite number of observations, u used

[2] A counting process (e.g. the number of exceedances occurring in a time period) with a rate $\lambda(t)$ is called a nonhomogeneous Poisson process. Specifically, let $T > 0$ be a real number and $N = \{N_t : t \in [0, T]\}$ be the process recording the number of values exceeding the threshold u until time T. The random variable N_t which represents the number of exceedances in the time interval $[0, t)$, $t \geq 0$, follows a Poisson distribution with rate function $\lambda(t)$.

for the estimation of the model is also finite. However, u should be high enough to ensure that the asymptotic theory, which underlies the GPD approximation, is valid and reliable. For the GPD, u is equivalent to selecting the sample fraction (k_n) considering only the k_n upper order statistics of the sequence $X_{(1)} \leq \cdots \leq X_{(k_n)} \leq \cdots \leq X_{(n)}$, where $n \gg k_n$ (see Leadbetter et al., 1983 for the characterization of k_n for specific distributions). In this case, as $n \to +\infty$, the $k_n \to +\infty$ such that $k_n/n \to 0$. Consequently, this implies that as the sample size n increases, the threshold u is also increased, at a faster rate though.

With these notations, when $\xi > 0$, $G_{\xi,\sigma}$ corresponds to a heavy-tailed distribution (Fréchet type distribution). When $\xi \to 0$, $G_{\xi,\sigma}(x) \to 1 - exp(-x/\sigma)$, which is an exponentially declining tail distribution and corresponds to a thin-tailed distribution (Gumbel type distribution). When $\xi < 0$, $G_{\xi,\sigma}$ corresponds to a distribution with a finite tail (Weibull type distribution).

For return distributions used in financial modeling, considering that the distribution of return exceedances is not exactly known, the asymptotic behavior of return exceedances can be used. Thus, we can easily compute the parameters of the limit distribution. For example, the normal distribution leads to a GPD with $\xi = 0$. The Student-t distributions and stable Paretian laws lead to a GPD with $\xi > 0$. Furthermore, the GPD can be extended to processes based on the normal distribution, such as autocorrelated normal processes, discrete mixtures of normal distributions and mixed diffusion jump processes. They all have thin tails and their domain of attraction is a GPD with $\xi = 0$. De Haan et al. (1989) showed that if returns follow a GARCH process, then the extreme return has a GDP with $\xi > 0$.

3.2 Multivariate modeling of extremes

The multivariate extreme value theory aims to quantify the strength of the dependence structure of extremes. In particular, multivariate extreme value analysis refers to the estimation of the joint probability of extreme events occurring together simultaneously. In the bivariate setting, a widely used nontrivial asymptotically dependent joint distribution is the logistic model. In what follows, first, we provide the general theoretical framework for modeling the dependence structure of extremes. Then, we discuss the statistical characterization of the bivariate tail dependence. Finally, we present the bivariate parametric models employed.

Modeling the dependence structure of extremes: A general framework

Consider a bidimensional vector of random variables denoted as $X = (X_1, X_2)$, with a bivariate distribution function F_X and an unknown dependence structure D_X. The bivariate return exceedances correspond to the vector of univariate return exceedances, defined with a bidimensional vector of thresholds $u = (u_1, u_2)$. Modeling the joint tail of distribution F_X entails two distinct aspects. The first one is the modeling of the tail of each marginal distribution (univariate aspect), while the second one is the modeling of dependence structure D_X within the joint tail (bivariate aspect).

In approaching the issue of modeling the dependence structure of vector X, the most commonly used heuristic method is through copulas. Copulas allow us to decompose the bivariate distribution into each margin and its joint behavior. In a general respect, a q-dimensional copula is a multivariate distribution in $[0,1]^q$ interval, while the marginal distributions - transformed in initial margins of distribution F_X - are uniform on the [0,1] interval (Sklar, 1959). In the study of bivariate dependence structure, q is equal to 2. By definition, under a common bivariate probability distribution F_X of vector Y of the transformed random variables $Y_1 = F_{X_1}(X_1)$ and $Y_2 = F_{X_2}(X_2)$, a copula function, is defined as follows:

$$C(y_1, y_2) = Pr(Y_1 \leq y_1, Y_2 \leq y_2) = F\left(F_{X_1}^{-1}(y_1), F_{X_2}^{-1}(y_2)\right), y_1, y_2 \in [0,1] \qquad (3)$$

The vector (Y_1, Y_2) is described by the same dependence structure as in (X_1, X_2). The initial function F_X can be expressed with a copula function as $F_X(x) = C\big(F_{X_1}(x_1), F_{X_2}(x_2)\big)$, which is an efficient transformation of F_X into C and into univariate marginal distribution functions F_{X_1} and F_{X_2} (see Reiss and Thomas, 2001).

The above discussion applies to the general case, though. As our aim is to model the dependence structure of extremes, the common practice is to assume that F_X is in the domain of attraction of a bivariate extreme value distribution (or equivalently an extreme value copula) with standard Fréchet margins. Fréchet margins is the appropriate statistical transformation for vector X when dealing with extremes with heavy-tailed behavior and tail dependence (Ledford and Tawn, 1997). This transformation removes the influence of marginal aspects such that the differences in distributions are ascribed to the dependence aspects alone. Therefore, unlike linear measures of dependency, the measures used in this study are no longer influenced by the form of the marginal distribution (Embrechts et al., 1997). Fréchet margins display either negative or positive dependence, defined as Fréchet lower and upper bound copulas, which correspond to the limit cases of extreme dependence (Yang et al., 2009). The Fréchet margins are given by $Z_1 = -1/\log F_{X_1}(X_1)$ and $Z_2 = -1/\log F_{X_2}(X_2)$ for X_1 and X_2, respectively, where F_{X_1} and F_{X_2} are the corresponding marginal distribution functions. Furthermore, $Pr(Z_1 > u) = Pr(Z_2 > u) \sim u^{-1}$ as $u \to \infty$ ($u_1 = u_2$ denoted by u for simplicity). Since variables Z_1 and Z_2 are on the same scale, this gives the same probability weight of extreme events for each variable. In practice, as the marginal distributions F_{X_1} and F_{X_2} are unknown, we replace their tails with the approximate tail form presented in Equation (2).

Statistical characterization of the bivariate tail dependence

In the study of the dependence structure of extremes, it is important to understand whether the components of a bivariate random vector exhibit asymptotic independence or asymptotic dependence. We now explain how these two classes of dependence structure allow for specific inferences about the joint occurrence of extreme events. In the asymptotic independence case, large values in both variables appear simultaneously less often than in the asymptotic dependence case. The fundamental difference between the two classes of dependence structure emerges as we approach the upper limits of the two variables. As for asymptotically independent variables, while one of the variables tends to its upper limit, the likelihood of the other being also close to its upper limit is equal to 0; yet, as for asymptotically dependent variables, the likelihood of both variables being close to their upper limit is always greater than 0. In other words, extreme events do not appear at the same time in the case of asymptotic independence and may appear at the same time in the case of asymptotic dependence. Moreover, the likelihood of extreme events occurring simultaneously increases as the dependence becomes stronger (in the special case of total dependence, extreme events always appear simultaneously). In technical terms, this phenomenon can be quantified by computing the following joint probability:

$$\lim_{u \to \infty} Pr(Z_1 > u | Z_2 > u) = \chi \tag{4}$$

where $\chi = 0$ for asymptotic independence, $\chi > 0$ for asymptotic dependence, and $\chi = 1$ for total dependence (see Sibuya, 1960; de Haan and Resnick, 1977; Ledford and Tawn, 1996).

Following from the previous discussion, asymptotic independence is reached in many cases. If the components of the distribution are independent, the exact independence of extremes is obtained as expected. But even if the components of the distribution are dependent, asymptotic independence can arise. A well-known example is the multivariate normal distribution (Galambos, 1978). Note that in practice, as the number of observations is finite (with the threshold u being also finite), it may be difficult to identify asymptotic independence because a finite sample exhibits *residual* dependence for any finite threshold. Consequently, although the limit in Equation (4) is equal to 0 in the asymptotic independence case, the joint probability is different from 0 before the limit is reached (see de Haan and Zhou, 2011).

Modeling the bivariate tail dependence

In the bivariate modeling, we transform our data into unit Fréchet margins for exceedances defined by the threshold u. Then, in line with Longin and Solnik (2001), and Poon et al. (2004), we consider asymptotically independent and dependent parametric models in order to study the issue of independence and dependence of extremes. In this case, the joint tail form of the bivariate distribution F_X plays a central role. As in the univariate case, the bivariate distribution F_X is not precisely known. Thus, the existing bivariate extreme value models are based on a bivariate asymptotic distribution (Resnick, 1987). As the threshold u tends to the upper endpoint x_{F_X}, the bivariate excess distribution F_X^u can only converge towards a distribution G_X characterized by a GPD for each margin and a dependence function D_X. The D_X must satisfy the following condition:

$$G_X(y_1, y_2) = exp\left(-D_X\left(-\frac{1}{y_1}, -\frac{1}{y_2}\right)\right), y_1, y_2 > 0 \qquad (5)$$

where $y_1 = -1/\log G_{\xi,\sigma}(x_1)$ and $y_2 = -1/\log G_{\xi,\sigma}(x_2)$. It is clear from Equation (5) that the bivariate asymptotic distribution G_X is not completely specified, as the shape of the dependence function D_X is not known, and therefore it has to be modeled.

As for the class of asymptotically independent models, the dependence function D_X is characterized by the following:

$$G_X(y_1, y_2) = exp\left(-\left(\frac{1}{y_1} + \frac{1}{y_2}\right)\right) \qquad (6)$$

To model the asymptotically independent components over a threshold u, Bortot et al. (2000) suggested the Gaussian model for Fréchet margins. This parametric model is a special case of the general tail model of Ledford and Tawn (1997) which includes both asymptotically independent and asymptotically dependent distributions. The Gaussian model for Fréchet margins is given by the following:

$$G_X(y_1, y_2) = \Phi_2\left(\Phi^{-1}\left(exp\left(-\frac{1}{y_1}\right)\right), \Phi^{-1}\left(exp\left(-\frac{1}{y_2}\right)\right); \rho\right), \rho < 1 \quad (7)$$

where Φ is the standard univariate normal distribution, while Φ_2 is a bivariate normal distribution with $\mu = (0, 0)$, having correlation matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The off-diagonal element corresponds to the correlation parameter which is considered the dependence parameter ρ. This model is computed from the multivariate Gaussian distribution with Fréchet margin transformation. As shown by Bortot et al. (2000), the parameter ρ captures the dependency of the bivariate normal distribution with transformed margins used to model the tails of the distribution. In the special case where the dependence structure over the whole distribution is normal, ρ is equal to the Pearson correlation.[3]

As for the class of asymptotically dependent models, we employ the logistic model proposed by the form of the dependence function of the Gumbel-Hougaard copula (see Gumbel, 1960; 1961; Hougaard, 1986) for Fréchet margins, as follows:

$$G_X(y_1, y_2) = exp\left(-\left(y_1^{-1/\alpha} + y_2^{-1/\alpha}\right)^{\alpha}\right) \quad (8)$$

This model contains the special cases of asymptotic independence and total dependence. It is parsimonious, as we only need one parameter to model the bivariate dependence structure of exceedances, i.e., the dependence parameter α with $0 < \alpha \leq 1$. The correlation of exceedances ρ (also called extreme correlation) can be computed from the dependence parameter α of the logistic model as follows: $\rho = 1 - \alpha^2$. The special cases where α is equal to 1 and α converges towards 0 correspond to asymptotic independence, in which ρ is equal to 0, and total dependence, in which ρ is equal to 1, respectively (Tiago de Oliveira, 1973).

The multivariate extreme value theory based on sufficiently large samples enables efficient inferences for the dependence structure of extremes. In practical applications, however, the limited number of exceedances, which is reduced as we move towards the endpoints of the distribution, induces a small sample estimation bias. Concerning the

[3] The Pearson correlation is a measure of the linear dependence between two variables; thus, it is not a sufficient measure of dependency when conditioning on extremes in the distribution tails. The Pearson correlation is estimated as the average of deviations of observations from the sample mean. The weight of observations in the distribution tails is equal to the weight of other observations. The Pearson correlation as a measure of extreme dependence has been called into question by Longin and Solnik (2001), Forbes and Rigobon (2002) and Embrechts et al. (2010), among others.

logistic model, further evidence provided by Huser et al. (2016) showed that the likelihood estimators (including the censored estimator) tend to overestimate the strength of dependence, as it becomes weaker in the case of finite samples. To reduce the estimation bias, we estimate a bias-corrected correlation of exceedances via a parametric bootstrap simulation. To this end, following Stephenson (2003), we simulate bivariate random samples from a bivariate extreme value distribution of a logistic type model.[4]

3.3 Statistical tests related to normality and dependency

We provide statistical tests based on the extreme correlation to study the issues of normality and dependency. From a practical point of view, these tests combine a statistical framework that can be used as a screening tool to deploy diversification possibilities between assets that are less vulnerable to non-diversifiable extreme risk. A statistical formalization of this problem provides a formal characterization of the level and degree of dependence between assets in periods of stress. As mentioned previously, for the two types of dependency (i.e., asymptotic independence and asymptotic dependence), the characteristics of returns are fundamentally different in terms of behavior, as these returns become more extreme. Although these two types permit dependency between moderately large values of two variables, extreme returns can occur at the same time only when asymptotic dependence exists. The knowledge of such a result is particularly important for portfolio risk management in order to control joint risk exposures.

First, we test if the observed extreme correlation corresponds to the case of normality. Any statistical deviation from normality is important in practice, as normality remains the standard assumption for modeling returns in asset management. Indeed, if the assumption of bivariate normality is violated, the use of normality could provide misleading results to describe portfolio risk under extreme market conditions, and then result in misguided diversification strategies. Second, we test if the observed extreme

[4] The maximum likelihood procedure has been shown to provide asymptotically unbiased estimates of the parameters of the model used. Of all the unbiased estimators, the maximum likelihood estimator has the smallest standard error (see Hosking and Wallis, 1987; van Gelder et al., 1999). In the case of fairly large samples, the maximization of the likelihood function with respect to the vector of the parameters allows us to numerically calculate reliable standard errors and confidence intervals (Coles et al., 2003). However, as noted by Koch et al. (1991), the maximum likelihood procedure does not always give unbiased estimates of the parameters. For example, in the case of small samples, there are significant computational problems inducing estimation bias (see e.g. Chaouche and Bacro, 2006). To avoid such types of bias, we apply a parametric bias-corrected approach based on the maximum likelihood procedure estimating a minimum variance unbiased estimator.

correlation corresponds to the case of independence or the case of total dependence. More importantly, we are interested in knowing if the joint occurrence of extreme returns in equity markets is significantly higher than that in the case of asymptotic independence. A statistical deviation from independence implies that the diversification benefits are limited, and even eliminated in the case of total dependence.

With respect to normality, we consider the following two cases: the asymptotic case and the finite-sample case. The asymptotic case corresponds to the theoretical case of an infinite number of observations, where the correlation of normal return exceedances is obtained as the limit when the threshold used to select extremes tends to infinity. In this case, the extreme correlation, denoted by ρ_{nor}^{asy}, is theoretically equal to 0 (Galambos, 1978). We also consider the finite-sample case because, as we discussed in subsection 3.2, it is not always possible to identify asymptotic independence with finite samples. This case corresponds to the empirical case of a finite number of observations, where the correlation of the normal return exceedances is computed with a given threshold u. In this case, the extreme correlation, denoted by $\rho_{nor}^{f.s.}(u)$, is greater than 0 (but decreasing towards 0 when the threshold tends to the endpoint of the distribution, that is $+\infty$ for the normal distribution).

- $H_0: \rho = 0$. We test the null hypothesis of *asymptotic* normality. That is, if the observed extreme correlation is equal to the extreme correlation in the asymptotic case obtained with a normal distribution of returns, ρ_{nor}^{asy}, which is equal to 0.
- $H_0: \rho = \rho_{nor}^{f.s.}(u)$. We test the null hypothesis of normality in the *finite-sample* case, that is, if the observed extreme correlation is equal to the extreme correlation in the finite-sample case obtained with a normal distribution of returns. In the finite-sample case, we compute $\rho_{nor}^{f.s.}(u)$ over a given finite threshold u by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The $\rho_{nor}^{f.s.}(u)$ is estimated via the logistic model given by Equation (8).

With respect to the issue of dependency, we consider the following two limit cases: independence and total dependence. The former case corresponds to an extreme correlation, ρ_{ind}, which is equal to 0, and the latter to an extreme correlation, ρ_{dep}, which is equal to 1.

- $H_0: \rho = 0$. We test the null hypothesis of *asymptotic independence* of extremes, that is, if the observed extreme correlation is equal to the extreme correlation obtained under asymptotic independence of extremes, ρ_{ind}, which is equal to 0.
- $H_0: \rho = 1$. We test the null hypothesis of *total dependence* of extremes, that is, if the observed extreme correlation is equal to the extreme correlation obtained under total dependence of extremes, ρ_{dep}, which is equal to 1.

4. Empirical results

This section presents our empirical results. First, we present the data, explain the data adjustments necessary to work with stationary time series and employ a data visualization procedure using nonparametric copulas to obtain preliminary evidence of the tail dependence patterns. Second, we present the parameter estimates of the bivariate model for the tail dependence structure. Third, we discuss the main findings of our study.

4.1 Data, data adjustments and data visualization

We analyze the tail dependence structure of international equity markets, i.e., Europe and the United States, vis-à-vis bitcoin and gold in a pairwise comparison. For the equity market in Europe (EU), we use the STOXX Europe 600 index, and for the equity market in the United States (US), we use the S&P 500 index. Both indices include the most heavily traded and liquid stocks with the largest market capitalization of their geographical zone.

Our empirical study covers the time-period from April 19, 2013 to April 17, 2018. Although bitcoin was first traded in 2010, we opt for the starting date of April 19, 2013 to avoid unreliable and spurious results due to the very low liquidity and resulting price variability of bitcoin during that period. From April 19, 2013, when bitcoin prices broke the $100 threshold for the first time, the impact of liquidity on market prices became less important. In our study, we consider weekly returns to avoid the time lag bias between the equity markets in Europe and the United States.[5] The data for the STOXX Europe 600 and S&P 500 indices, bitcoin and gold come from Bloomberg.[6]

For each time series of returns, we apply a data adjustment procedure based on the work of Gallant et al. (1992) to remove trends and apply the work of McNeil and Frey (2000) to take into account heteroskedasticity due to clusters. Thus, we limit the sample bias observed for serially correlated and clustered data. We describe in detail our data adjustment procedure in Appendix 2.

Finally, as a preliminary analysis, we use nonparametric copulas to provide a graphical visualization of the dependence patterns in our data. In Appendix 3, we present the statistical procedure based on surface plots obtained with a kernel-type copula density estimator. Following our four-step research strategy, we find graphical evidence of strong tail dependence between the European and US equity markets during stock market crashes and weak tail dependence between equity markets and bitcoin or gold both in bear and bull markets. We also find a weak tail dependence between bitcoin and gold. These results offer early recognition of the role of bitcoin and gold as diversifiers. Next, we quantify this preliminary evidence of the tail dependencies with parametric copulas.

4.2 Estimation of the parameters of the bivariate model

We now discuss the estimation of the parameters of the bivariate model for the tail dependence structure. In line with our four-step research strategy, we present our empirical results in four sets of tables. We adopt this description plan in order to keep the paper consistent in terms of the presentation of the results over the sections.

Table 1 refers to the bivariate tail dependence structure between the equity markets in Europe and the United States (EU/US). Table 2 refers to the bivariate tail dependence structure between each equity market and bitcoin, i.e., Table 2A for Europe

[5] The US equity market is the last to close on a trading day compared to the European one. Therefore, due to the time lag, it is likely that extreme price movements in the US equity market affect the European equity market within the next days. Rapach et al. (2013) pointed out the leading role of the US equity market as they found strong evidence that lagged US returns have significant predictive power for non-US returns.

[6] For bitcoin prices, we use the bitcoin Bloomberg Index (Code: XBT). Using a sophisticated pricing algorithm, Bloomberg computes the bitcoin index as a weighted-average of mid-prices derived from bid and ask quotes from multiple approved Cryptocurrency Pricing Sources (approved exchanges following anti-money laundering (AML) and know your customer (KYC) policies, and providing an organizational and/or ownership chart). In practice, the exchanges selected by Bloomberg are mainly located in Europe and North America that are integrated geographical zones with the same playing field in terms of fiscal and financial regulations.

and bitcoin (EU/BTC) and Table 2B for the United States and bitcoin (US/BTC). Table 3 refers to the bivariate tail dependence structure between each equity market and gold, i.e., Table 3A for Europe and gold (EU/Gold) and Table 3B for the United States and gold (US/Gold). Table 4 refers to the bivariate tail dependence structure between bitcoin and gold (BTC/Gold). Overall, we study the following pairs among international equity indices, bitcoin and gold, namely, EU/US, EU/BTC, US/BTC, EU/Gold, US/Gold and BTC/Gold. For each table, Panel A refers to the negative return exceedances in the left tail of the distribution and Panel B to the positive return exceedances in the right tail.

We provide maximum likelihood estimates of the parameters of the bivariate extreme distribution for both the fixed and optimal thresholds. We define the fixed threshold (i.e., selected before fitting) with tail probability levels across the entire range of the left and right distribution tails of returns as follows: 50%, 40%, 30%, 20%, 10% and 5%. For each pair, we use the same value of probability level p to define the return exceedances in each time series. We also compute the optimal thresholds by following the procedure described in Appendix 4. As explained by Jansen and de Vries (1991), optimal thresholds optimize the trade-off between inefficiency and sample bias. A low threshold value induces a significant estimation bias due to the observations not belonging to the distribution tails considered as exceedances. A high threshold value leads to inefficiency with increasing standard errors due to the reduced size of the estimation sample. We report the estimates obtained with an optimal threshold on the last line of each panel. In total, the following parameters are reported: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the dependence between extreme returns, the correlation of return exceedances ρ, and in the last columns the Wald tests of the statistical hypotheses presented in subsection 3.3. We also provide the standard errors of the estimates in parentheses, whereas the p-values for the corresponding Wald tests are in brackets.

A graphical representation of our estimates in Tables 1-4 is also given in Figures 1-4, corresponding to each step of our research strategy. In these figures, we depict the evolution of the correlation of return exceedances moving towards the distribution tails. The value of the tail probability p is used to define return exceedances associated with a threshold u. These figures also graphically capture the potential asymmetry between negative and positive return exceedances in the left and right distribution tails. The solid line represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled via the logistic model. The dotted line represents the theoretical correlation in the finite-sample case between the simulated normal return exceedances, assuming a bivariate normal return distribution with parameters equal to the empirically observed means and covariance matrix of returns.

4.3 Main empirical results

We now present our main empirical results about the estimation of the parameters of the bivariate model for the tail dependence structure. We follow our four-step research strategy highlighting the most important findings in each step and providing key comparisons.

Step 1: Equity markets

Table 1 refers to the bivariate tail dependence structure between the equity markets in Europe and the United States (EU/US). We confirm the stylized fact of the behavior of equity markets during extremely volatile periods. We find that the tail dependence increases in bear markets and decreases in bull markets. Such evidence indicates that traditional dependence measures used to model extreme co-movements in equity markets are inadequate and could lead to inaccurate portfolio diversification strategies. In one of the first works on this matter, Longin and Solnik (2001) found similar results between the main European equity markets (France, Germany, the United Kingdom) and the US equity market. The level of extreme correlation during stock market crashes is even higher in our study, which uses a more recent time period, i.e., 0.878 *vs* 0.571 for the correlation for negative return exceedances at optimal threshold levels. The level of extreme correlation during stock market booms is also higher in our study, i.e., 0.384 *vs* 0.140 for the correlation for positive return exceedances. Chabi-Yo et al. (2018) also found a general tendency for stronger asymptotic dependence in the left tail than in the right tail of the return distribution in the recent period, reflecting more integrated international stock markets.

More specifically, for negative return exceedances (Panel A), we observe that the correlation of return exceedances ρ slightly increases across the left tail of the distribution. It is equal to 0.888 for p = 50% and 0.890 for p = 5%. The correlation of the return exceedances ρ at the optimal thresholds is equal to 0.878. With respect to asymptotic normality, we reject the null hypothesis, i.e., H_0: $\rho = \rho_{nor}^{asy}$, as the first Wald test shows, across the entire range of the left distribution tail. The value of this test is equal to 26.641 for p = 50% and 203.487 for p = 5%. At the optimal thresholds, it is equal to 63.419 and leads to a strong rejection of the null hypothesis. With respect to normality in the finite-sample case, we also reject the null hypothesis, i.e., $H_0: \rho = \rho_{nor}^{f.s.}(u)$, moving to the left endpoint of the distribution for tail probability levels lower than 20%, as the second Wald test suggests. The value of this test is equal to 0.694 for p = 50%, 2.152 for p = 20%, and 3.726 for p = 5%. At the optimal thresholds, it is equal to 3.130 and leads to the rejection of the null hypothesis. Hence, the probability of large losses occurring at the same time in the two markets is significantly higher than under the hypothesis of bivariate normality, let alone in the case of asymptotic normality. With respect to the asymptotic independence of extremes, as expected, we reject the null hypothesis, i.e., $H_0: \rho = 0$, as the first Wald test shows, across the entire range of the left distribution tail. With respect to the total dependence of extremes, we reject the null hypothesis: $H_0: \rho = 1$ for all threshold values. The value of this test is equal to 3.256 for p = 50% and 25.008 for p = 5%. At the optimal thresholds, it is equal to 8.678 and leads to a strong rejection of the null hypothesis. Although the tests suggest a strong rejection of the null hypothesis, the correlation of the return exceedances ρ is still high enough to provide diversification benefits during periods of extreme price volatility in equity markets.

As for positive return exceedances (Panel B), we observe that the correlation of return exceedances ρ declines across the right tail of the distribution. It is equal to 0.864 for p = 50% and 0.521 for p = 5%. The correlation of return exceedances ρ at the optimal thresholds is equal to 0.384. With respect to asymptotic normality, we reject the null hypothesis, i.e., $H_0: \rho = \rho_{nor}^{asy}$, as the first Wald test shows, across the entire range of the right distribution tail. The value of this test is equal to 24.194 for p = 50% and 17.260 for p = 5%. At the optimal thresholds, it is equal to 61.206 and leads to a strong rejection of the null hypothesis. Furthermore, unlike negative return exceedances, with respect to normality in the finite-sample case, we cannot reject the null hypothesis, i.e., $H_0: \rho = \rho_{nor}^{f.s.}(u)$, moving to the right endpoint of the condition distribution for all values of u under consideration, as the second Wald test suggests. The value of this test is equal to 0.169 for p = 50% and 0.548 for p = 5%. At the optimal thresholds, it is equal to 0.199. With respect to the asymptotic independence of extremes, we reject the null hypothesis, i.e., $H_0: \rho = 0$, as the first Wald test shows, across the entire range of the right distribution tail. With respect to the total dependence of extremes, we reject the null hypothesis, i.e.,

$H_0\colon \rho = 1$, of total dependence in all cases. The value of this test is equal to 27.155 for p = 50% and 32.592 for p = 5%. At the optimal thresholds, it is equal to 90.987 and leads to a strong rejection of the null hypothesis.

The asymmetry between negative and positive return exceedances is confirmed by Figure 1, which refers to the bivariate tail dependence structure between the European and US return exceedances (EU/US). As shown in Figure 1, the correlation of negative return exceedances is always greater than the correlation of positive return exceedances.

Step 2: Equity markets and bitcoin

Tables 2A and 2B refer to the bivariate tail dependence structure between each equity market and bitcoin, i.e., Europe and bitcoin (EU/BTC) and the US and bitcoin (US/BTC). In this step, we combine each equity market with bitcoin to assess the potential diversification benefits of bitcoin in times of stress. Contrary to Step 1 for equity markets, we find that the tail dependence between each equity market and bitcoin decreases in both bear and bull markets. Thus, bitcoin could provide significant diversification benefits to investors against adverse price market movements. Given that high levels of tail dependence indicate greater risk exposures when a crisis occurs, investors should identify positions that present low levels of tail dependence to protect their positions. As shown by the results, bitcoin can play the role of the diversifier.

More specifically, as for Table 2A for the pair EU/BTC, we observe that the dependency declines moving towards the distribution tails. Regarding negative return exceedances (Panel A), the correlation of return exceedances ρ is equal to 0.477 for p = 50% and 0.019 for p = 5%. Compared to Step 1, this means that potential diversification benefits can better be achieved by holding bitcoin with the European equity position. Regarding positive return exceedances (Panel B), the correlation of return exceedances is equal to 0.609 for p = 50% and 0.084 for p = 5%. Furthermore, we cannot reject the null hypothesis of normality, i.e., $H_0\colon \rho = \rho_{nor}^{f.s.}(u)$, in the finite-sample case, in which the correlation of the return exceedances is statistically equal to the correlation obtained with a bivariate normal distribution. This means that the probability of large losses occurring at the same time in the two markets is not significantly higher than under the hypothesis of bivariate normality. There is also strong evidence, that this probability decreases as we move further in the tails of the distribution. For Table 2B for the pair US/BTC, a similar conclusion is drawn. The dependency declines moving towards the distribution tails. Thus, we cannot reject the null hypothesis that the correlation of return exceedances follows a bivariate-normal distribution in most fixed thresholds in both distribution tails. Regarding negative return exceedances (Panel A), the correlation of return exceedances is equal to 0.547 for p = 50% and 0.123 for p = 5%. Regarding positive return exceedances (Panel B), the correlation of return exceedances is equal to 0.599 for p = 50% and 0.200 for p = 5%.

Figures 2A and 2B depict the bivariate tail dependence structure between each equity market and bitcoin (EU/BTC and US/BTC). Unlike Figure 1 for equity markets alone, we observe that the extreme correlation for both negative and positive return exceedances decreases when we go further into the tails. Moreover, this statistical behavior appears to be symmetric.

Step 3: Equity markets and gold

Tables 3A and 3B refer to the bivariate tail dependence structure between each equity market and gold, i.e., Europe and gold (EU/Gold) and the US and gold (US/Gold). In this step, we combine each equity market with gold to confirm its well-known diversification benefits in times of extreme price volatility. Similarly, as in Step 2, we find that the tail dependence between each equity market and gold decreases in bear markets. Therefore, it confirms the status of gold as a safe haven.

More specifically, as for Table 3A for the pair EU/Gold, we find that the dependency declines moving towards the distribution tails. We observe quite similar bivariate patterns among the dependency of return exceedances between the pairs of each equity market and bitcoin or gold. The correlation of return exceedances is equal to 0.522 for p = 50% and 0.060 for p = 5% for negative return exceedances (Panel A). Regarding positive return exceedances (Panel B), the correlation of return exceedances is equal to 0.606 for p = 50% and 0.372 for p = 5%. For Table 3B for the US/Gold pair, a similar conclusion is obtained. The dependency declines moving towards the distribution tails. Furthermore, for most fixed thresholds in both distribution tails, we cannot reject the null hypothesis of normality in the finite-sample case, i.e., $H_0\colon \rho = \rho_{nor}^{f.s.}(u)$, in which the correlation of return exceedances is statistically equal to the correlation obtained with a bivariate normal distribution. Regarding the negative return exceedances (Panel A), the correlation of the return exceedances is equal to 0.558 for p = 50% and 0.089 for p = 5%. Regarding the positive return exceedances (Panel B), the correlation of the return exceedances is equal to 0.614 for p = 50% and 0.259 for p = 5%.

Figures 3A and 3B depict the bivariate tail dependence structure between each equity market and gold (EU/Gold and US/Gold). In contrast with Figure 1 for equity markets only, the extreme correlation for both negative and positive return exceedances decreases when we go further into the tails. As in the case of bitcoin (Figures 2A and 2B), this statistical behavior also appears to be symmetric.

Step 4: Bitcoin and gold

Table 4 refers to the bivariate tail dependence structure between bitcoin and gold (BTC/Gold). In this step, we consider a position in bitcoin and gold only to observe if both assets can provide diversification benefits against extreme market downturns at the same time. We find that the tail dependence between bitcoin and gold decreases in both bear and bull markets. Thus, this indicates that both bitcoin and gold can be used together as complementary diversifiers in times of turbulence of financial markets.

As for Table 4, the dependency also declines moving towards the distribution tails. We cannot reject the null hypothesis of normality, i.e., $H_0\colon \rho = \rho_{nor}^{f.s.}(u)$, in the finite-sample case, in which the correlation of return exceedances is statistically equal to the correlation obtained with a bivariate normal distribution for most fixed thresholds in both distribution tails. More specifically, the correlation of negative return exceedances (Panel A) is equal to 0.520 for p = 50% and 0.083 for p = 5%. The correlation of positive return exceedances (Panel B) is equal to 0.590 for p = 50% and 0.106 for p = 5%.

Figure 4 depicts the bivariate tail dependence structure between bitcoin and gold (BTC/Gold). We observe that the extreme correlation for both the negative and positive return exceedances decreases when we go further into the tails.

5. Bitcoin *vs* gold

In this section, we evaluate the diversification benefits of using bitcoin and gold together during periods of stress in equity markets. We first discuss the general economic backdrop of bitcoin and gold. We then compare the extreme correlation between equity markets and bitcoin or gold. Finally, we assess the joint potential of bitcoin and gold as diversifiers for equity positions.

5.1 Economic backdrop

Bitcoin and gold are two fundamentally different assets in several respects. On the one hand, bitcoin is a decentralized digital currency without physical form and apparently much newer than gold. At first blush, bitcoin exhibited a very shadowy background related to theft, fraud and criminal activity (see e.g. Gandal et al., 2018). It also appears that bitcoin is free of sovereign risk, as it is independent from regulatory authorities, central banks and governments.[7] On the other hand, gold has a very good reputation having the trust of the financial community and displaying the willingness of investors to hold positions in it. Investors continue to believe that gold can be used as a hedge against a declining US dollar and rising inflation. It is also one of the first forms of money and thus keeps its purchasing power in the long run. But apart from their differences, bitcoin and gold also present some key similarities. Primarily, they are both nonproductive assets and speculative investments, as they do not produce future cash flows. Interestingly enough, bitcoin and gold have both been characterized as safe havens. However, is really bitcoin as *good* as gold for investors?

Gold has long been identified as a safe haven with an indisputable track record spanning more than 5,000 years of history, as well as being a physical store of value across human civilizations. Although there is no theory explaining why gold is commonly referred to as a safe haven - as pointed out by Baur and Lucey (2010) -, it has earned its well-deserved reputation over various past economic disasters. Many financial advisors regularly recommend that investors should hold from 10% to 30% of their long-term portfolios in gold or gold-related positions. With the advent of bitcoin, a debate has started among practitioners covered in the financial press (see Mackintosh, 2017; Rob, 2018; Somerset Webb, 2018; Taplin, 2018, among others). The debate has centered principally on the issue of whether bitcoin has similar appealing properties to gold or not. Hitherto the search “bitcoin + gold” resulted in an overwhelming number of 335,000,000 results in Google (Google.com accessed on Nov 28, 2019). Not surprisingly, a number of professionals have indicated full or qualified support in favor of bitcoin. Such arguments, however, stand as anecdotal evidence, while relevant studies on this issue are relatively scarce. To contribute to the current debate on the role of the two assets, we study in a rigorous way whether bitcoin has an advantage over gold in terms of diversification benefits during extremely volatile periods. Given that such market conditions are a primary concern for investors, we base our contribution on the multivariate extreme value

[7] Bitcoin uses the blockchain technology, which ensures that any transaction is unique, and users can complete transactions without any intervention from regulatory authorities, central banks and governments. See Yermack (2017) for additional information regarding blockchains.

theory, which constitutes a formal statistical tool to model the dependence structure of extremes. Besides this, it can also capture nonlinearities that are more relevant to financial crashes. Going a step further, we also wonder if both bitcoin and gold can be useful together as complementary diversifiers.

As our paper focuses on diversification benefits, we mainly study market risk through the distribution of returns (via the tail dependence). Note that beyond market risk, there are other relevant types of risk to consider before selecting an asset, mainly in crisis situations, such as liquidity risk (with higher transaction costs), regulatory and governance risk (changes of rules), political risk (ban of cryptocurrencies by some countries) and operational risk (failure of exchanges due to hacking). The size of the market is of importance, as well. Taking all these into consideration, the evidence provided here for the role of bitcoin in asset allocation and portfolio risk management should be assessed by investors on a regular basis. Bitcoin has a much shorter history than gold and some "new" types of idiosyncratic risk are still under investigation or need to be further studied. Future work should assess in practice their importance compared to gold, but we do not explore such types of risk here.

5.2 Diversifiers for equity markets: Bitcoin or gold?

Table 5 compares the results obtained in Steps 2 and 3 of our research strategy. Panel A reports the extreme correlation between the European equity market and bitcoin $\rho^{EU/BTC}$, and between the European equity market and gold $\rho^{EU/Gold}$. Panel B reports the extreme correlation between the US equity market and bitcoin $\rho^{US/BTC}$ and between the US equity market and gold $\rho^{US/Gold}$. In each panel, we test the following null hypotheses: $H_0: \rho^{EU/BTC} = \rho^{EU/Gold}$ and $H_0: \rho^{US/BTC} = \rho^{US/Gold}$, with a Wald test, to assess the potential advantages of bitcoin and gold in terms of diversification benefits during downside market conditions.

As for the European equity market (Panel A) for negative return exceedances, we cannot reject the null hypothesis of equality, i.e., $H_0: \rho^{EU/BTC} = \rho^{EU/Gold}$, at optimal threshold levels, as the value of the Wald test is equal to 0.482. For positive return exceedances, we also cannot reject the null hypothesis, as the value of the Wald test is equal to 0.785. For the US equity market (Panel B) for negative return exceedances, we cannot reject the null hypothesis of equality, i.e., $H_0: \rho^{US/BTC} = \rho^{US/Gold}$, at optimal threshold levels, since the value of the Wald test is equal to 0.886. For positive return exceedances, we also cannot reject the null hypothesis, as the value of the Wald test is equal to 0.872.

Figure 5A depicts the extreme correlation between the European equity market and bitcoin and between the European equity market and gold. Figure 5B depicts the extreme correlation between the US equity market and bitcoin and between the US equity market and gold. The differences in the extreme correlation between the pairs under consideration are mostly statistically nonsignificant. Put differently, this implies that the probability of large losses occurring at the same time both in European or US equity markets and bitcoin is not significantly different (be it lower or higher) from the one observed when combining each equity market with gold.

Overall, considering a separate addition of bitcoin or gold in an equity position, our findings show that an equity position including gold does not have a significant advantage over an equity position including bitcoin in times of extreme price volatility. Although advantages and disadvantages can be found for both speculative assets, our extreme value analysis contributes to the debate on which asset is superior and why; our approach is not based on philosophical premises of progressivists and conservationists, yet it is based on stringent data analysis with a suitable statistical tool for portfolio risk management. Regardless of some clear advantages that gold may have over bitcoin, we demonstrate empirically that the precious metal has a "real" competitor at least for the choice of diversifier against adverse price movements in equity markets. Consequently, from the perspective of diversification benefits, we conclude that bitcoin can be considered the new digital gold.

5.3 Joint diversifiers for equity markets: Bitcoin and gold?

The findings obtained in Step 4 of our research strategy reveal clear evidence that both bitcoin and gold can be useful together in times of turbulence in financial markets. We find a decreasing correlation between the bitcoin and gold return exceedances by going further into the left and right tails. We observe very low correlation levels, as follows: the correlation of negative return exceedances is equal to 0.054 at the optimal threshold levels, while the correlation of positive return exceedances is equal to 0.024. Both assets, i.e., bitcoin and gold, could then be added to a position in equity markets to provide additional diversification benefits.

Although we previously mentioned that gold has a "real" competitor, in fact bitcoin and gold should not be considered as substitute assets in a portfolio. Actually, they should be used jointly as complementary diversifiers against extreme price fluctuations. The general outlook suggests that the portfolio risk declines as a higher number of assets in it increases (Statman, 1987), while diversification benefits tend to become larger when these assets are less correlated (see Longin and Solnik, 1995; Errunza et al., 1999; Driessen and Laeven, 2007; Kalemli-Ozcan et al., 2013, among others). Against this backdrop, we introduce the new concept of *diversification in diversifiers* (hereafter D-*in*-D) which refers to the use of several diversifiers with low levels of extreme correlation. D-*in*-D can also reduce the various types of risk exposures discussed in subsection 5.1. As a consequence, in terms of proper portfolio construction, regarding the question "bitcoin or gold", our empirical analysis provides promising evidence that including both bitcoin *and* gold is the best answer to diversify a position in equity markets when extreme price movements happen. Such evidence increases the future utility of bitcoin for the financial community. Below, we further evaluate this concept by performing several robustness checks.

6. Robustness

In this section, having obtained strong results in favor of bitcoin as a diversifier, we perform several robustness checks to validate the answer to the following question: "Is bitcoin the new digital gold?". First, we extend our modeling approach to the extreme value models of the logistic family to check whether the behavior of tail dependence holds in a more generalized framework. Second, we expand our empirical study to other countries of international financial interest. Third, we also expand our empirical study to other cryptocurrencies with increasing popularity. Fourth, we study whether our results about extreme correlation are stable over time.

6.1 Extended bivariate modeling of extremes

We previously used the logistic model to estimate the tail dependence in the distribution of asset returns and to assess the diversification benefits during extremely volatile periods. We now extend our baseline statistical model by fitting different models from the logistic family. The models under consideration are as follows: (i) the asymmetric logistic model (an extension of the logistic model allowing for asymmetry and nonexchangeability),[8] (ii) the negative logistic model (a variant of the logistic model as a survival model) and (iii) the asymmetric negative logistic model (an extension of the negative logistic model allowing for asymmetry). Then, we provide a comparative discussion of the results obtained with these different models.

Bivariate extreme value models

The asymmetric logistic model proposed by Tawn (1988) for Fréchet margins is given by the following:

$$G_X(y_1, y_2) = exp\left(-\frac{1-\theta_1}{y_1} - \frac{1-\theta_2}{y_2} - \left(\left(\frac{y_1}{\theta_1}\right)^{-1/\alpha} + \left(\frac{y_2}{\theta_2}\right)^{-1/\alpha}\right)^{\alpha}\right) \quad (9)$$

where $0 < \alpha \leq 1$, $0 \leq \theta_1 \leq 1$ and $0 \leq \theta_2 \leq 1$. Asymptotic independence is obtained when $\alpha = 1$ with $\theta_1 = 0$ or $\theta_2 = 0$. Total dependence is obtained when $\theta_1 = \theta_2 = 0$ and $\alpha \to 0$. Our baseline model given by Equation (8) corresponds to the following special case: $\theta_1 = \theta_2 = 1$.

The negative logistic model proposed by Galambos (1975) for Fréchet margins is given by the following:

$$G_X(y_1, y_2) = exp\left(-y_1 - y_2 + ({y_1}^{-\alpha} + {y_2}^{-\alpha})^{-1/\alpha}\right) \quad (10)$$

with dependence parameter $\alpha > 0$. Asymptotic independence is obtained when $\alpha \to 0$, while total dependence is obtained when $\alpha \to +\infty$.

The asymmetric negative logistic model proposed by Joe (1990) for Fréchet margins is given by the following:

$$G_X(y_1, y_2) = exp\left(-y_1 - y_2 + ((\theta_1 y_1)^{-\alpha} + (\theta_2 y_2)^{-\alpha})^{-1/\alpha}\right) \quad (11)$$

with dependence parameter $\alpha > 0$ and asymmetric parameters θ_1 and θ_2, where $0 \leq \theta_1$, $\theta_2 \leq 1$. When $\theta_1 = \theta_2 = 1$, the asymmetric negative logistic model is equivalent to the negative logistic model. The independence is obtained, as one of the parameters α, θ_1 or θ_2 tends to 0, while total dependence is obtained when $\theta_1 = \theta_2 = 1$ and $\alpha \to +\infty$.

[8] With the term nonexchangeability, we refer to the case of more asymmetric copulas as we model more dependent relations between random variables (see Nelson, 2007; Durante, 2009, among others).

The bivariate extreme value models presented above have the general representation form, as described in Equation (5). This equation can also be written as follows:

$$G_X(y_1, y_2) = exp\left(-\left(\frac{1}{y_1} + \frac{1}{y_2}\right) A\left(\frac{y_2}{y_1 + y_2}\right)\right) \quad (12)$$

where the $A(\cdot)$ is known as the Pickands depedence function, whereas A: [0,1] → [1/2,1] is a convex function, satisfying $\max\{t, 1 - t\} \leq A(t) \leq 1$ and $A(0) = A(1)$ =1. One of the key points resulting from the study of Pickands (1981), based on de Haan and Resnick's (1977) findings, is that it allows us to uniquely characterize a bivariate extreme value model by means of a finite measure defined on the unit simplex. More precisely, the function $A(\cdot)$ and any bivariate extreme value model are linked by the following relation: $A(\omega) = D_X(y_1, y_2)/(y_1^{-1} + y_2^{-1})$, where $\omega = y_2/(y_1 + y_2)$. For a bivariate parametric dependence function $A(\cdot)$ defined by threshold u corresponding to the q-quantile, the strength of quantile dependence is measured by the parameter $\chi(q)$. The quantile dependence parameter $\chi(q)$ is approximately equal to $2(1 - A(1/2))$, where $0 \leq \chi(q) \leq 1$ and $q \in (0,1)$. The special cases where $\chi(q)$ is equal to 1 (for all q) and $\chi(q)$ is equal to 0 (for all q) correspond to asymptotic independence and total dependence, respectively (Coles et al., 1999).

The quantile dependence parameter $\chi(q)$ is closely related to the dependence parameter α, previously used as a measure of the strength of tail dependence. However, the relation between the dependence parameter α and the quantile dependence parameter $\chi(q)$ varies according to the extreme value model. For example, for the logistic model, asymptotic independence is obtained when $\alpha \to 1$, while for the negative logistic model, asymptotic independence is obtained when $\alpha \to 0$. Hence, in this subsection, we consider the quantile dependence parameter $\chi(q)$ to compare the dependence measure across all models of the logistic family used in this study.

Comparative results across bivariate extreme value models

Table 6 refers to the quantile dependence parameter estimates for the four models presented above. For each model, we provide the maximum likelihood estimate for the quantile dependence parameter $\chi(q)$ and the Akaike information criterion (AIC) at optimal thresholds. The quantile q at optimal thresholds is defined as the corresponding tail probability p for the negative return exceedances and $(1 - p)$ for the positive return exceedances. The minimum value of the AIC across the four models is highlighted in bold. Panel A refers to negative return exceedances in the left tail of the distribution, and Panel B refers to positive return exceedances in the right tail.

We present the empirical results following our four-step research strategy. More specifically, regarding Step 1, which refers to the quantile dependence between European and the US equity markets (EU/US), we find that the level of dependency is relatively high in bear markets and significantly lower in bull markets across all bivariate extreme value models. Regarding Panel A and the negative return exceedances, the quantile dependence parameter $\chi(q)$ is equal to 0.764 for the logistic model, 0.533 for the asymmetric logistic model, 0.769 for the negative logistic model and 0.476 for the asymmetric negative logistic model. Regarding Panel B and the positive return exceedances, the quantile dependence parameter $\chi(q)$ is equal to 0.274 for the logistic model, 0.304 for the asymmetric logistic model, 0.338 for the negative logistic model and 0.345 for the asymmetric negative logistic model. Furthermore, the AIC is at the minimum for the negative logistic model. Regarding Step 2, which refers to the quantile dependence between equity markets and bitcoin (EU/BTC and US/BTC), we find a weak level of dependency between equity markets and bitcoin across all bivariate extreme value models, both in bear and bull markets. Regarding Step 3, which refers to the quantile dependence between equity markets and gold (EU/Gold and US/Gold), a similar result is obtained as in the previous step. Finally, regarding Step 4, which refers to the quantile dependence between bitcoin and gold (BTC/Gold), we also find a weak level of dependency both in bear and bull markets.

Further supporting our earlier findings reached by the logistic model, by considering several extensions within the logistic family, we conclude that our findings are robust to the choice of the model used to study the tail dependence structure.

6.2 Expansion to other international equity markets

We now examine whether our empirical results hold for other international equity markets of financial interest, as follows: two Asian equity markets (China and Japan) and three European equity markets (France, Germany and the United Kingdom). Regarding the Asian equity markets, we consider the SSE 180 index for China and the Nikkei 225 index for Japan. Regarding the European equity markets, we consider the CAC 40 index for France, the DAX 30 index for Germany and the FTSE 100 index for the United Kingdom (UK). All these indices, which include heavily traded and liquid stocks, are important benchmarks for the asset management industry. The data for the SSE 180, Nikkei 225, CAC 40, DAX 30 and FTSE 100 indices come from Bloomberg.

Similar to Step 2 and 3 of our research strategy, we analyze the tail dependence structure of these international equity markets vis-à-vis bitcoin and gold in a pairwise comparison. We repeat our analysis for the same time period (from April 19, 2013 to April 17, 2018) by applying in each new time series the data adjustment procedure described in Appendix 2.

We present the extreme correlations between each equity market and bitcoin and between each equity market and gold in Figure 6. More specifically, Figures 6A and 6B refer to the Asian equity markets, that is, the Chinese equity market and the Japanese equity market, respectively. Figures 6C, 6D and 6E refer to the European equity markets, that is, the French equity market, the German equity market and the UK equity market, respectively.

As shown in Figure 6, the correlation of return exceedances between equity markets and bitcoin declines moving towards the distribution tails for all equity markets. We observe similar bivariate dependence patterns between equity indices and gold. More specifically, considering a separate addition of bitcoin in an equity position, our findings show that an equity position including gold does not have any significant advantage against an equity position including bitcoin in periods of extreme volatility, in five additional international equity markets.

In conclusion, considering other major international equity markets, we also show that bitcoin can be regarded as a strong candidate for offering diversification benefits in a world-wide context.

6.3 Expansion to other cryptocurrencies

We now examine whether our empirical results hold for other cryptocurrencies. We consider the market-weighted cryptocurrency index, the so-called CRIX, which was developed by Trimborn and Härdle (2018), as a benchmark for the cryptocurrency market. We find a correlation between bitcoin and CRIX equal to 0.772, a high value as expected, since bitcoin has the largest weight in the CRIX index by far. Unsurprisingly, by applying our four-step research strategy with CRIX instead of bitcoin, we find similar results. To remove the effect of bitcoin, we then build on the CRIX index to construct another market-weighted cryptocurrency index that excludes bitcoin. We name this modified CRIX index as m-CRIX.

Figure 7 depicts the results obtained from our research strategy using the m-CRIX index. We repeat our analysis for the same time-period (from April 19, 2013 to April 17, 2018) by applying the data adjustment procedure described in Appendix 2 to the m-CRIX index. We present the new estimates of correlation of return exceedances in Figure 7. More specifically, Figure 7A refers to bitcoin and the m-CRIX index; Figure 7B refers to the European equity market and m-CRIX index; Figure 7C refers to the US equity market and m-CRIX, and Figure 7D refers to the m-CRIX and gold.

As shown in Figure 7, the following results are obtained: first, we observe that the correlation of the return exceedances between bitcoin and m-CRIX increases in bear markets and decreases in bull markets. Thus, there is little incentive to use bitcoin and other cryptocurrencies together in times of turbulence of financial markets. Second, the correlation of the return exceedances between equity indices and m-CRIX declines moving towards the distribution tails. We observe similar bivariate dependence patterns between the equity markets and m-CRIX and between the equity markets and bitcoin, as in Step 2 of our research strategy. Therefore, considering a separate addition of m-CRIX in an equity position, our findings show that other cryptocurrencies could also provide significant diversification benefits to investors during periods of extreme price volatility. Finally, we also observe that the correlation of the return exceedances between m-CRIX and gold decreases in both bear and bull markets, as in Step 4 of our research strategy. This clearly indicates that both m-CRIX and gold can be used together when financial markets are facing times of turbulence.

In short, by considering other cryptocurrencies as diversifiers, we also conclude that they can also provide potential diversification benefits for an equity position. This is a promising result for the role that not only bitcoin but also cryptocurrencies can play in general in the new digital era.

6.4 Stability over time of extreme correlation

We now examine whether our empirical results are stable over time. Once again, we present the empirical results based on our four-step research strategy. Figure 8 depicts the extreme correlation estimates over time using a 1-step ahead rolling-estimation window for the negative return exceedances, while Figure 9 presents the estimates for the positive return exceedances. Figure 8A (9A) refers to equity markets (Step 1), i.e., the European and US equity markets (EU/US). Figures 8B (9B) and 8C (9C) refer to equity markets and bitcoin (Step 2), i.e., the European equity market and bitcoin (EU/BTC) and the US equity market and bitcoin (US/BTC), respectively. Figures 8D (9D) and 8E (9E) refer to equity markets and gold (Step 3), i.e., the European equity market and gold (EU/Gold) and the US equity market and gold (US/Gold), respectively. Figure 8F (9F) refers to bitcoin and gold (Step 4), i.e., bitcoin and gold (BTC/Gold).

More specifically, as for Step 1 and Figures 8A and 9A, we find that the extreme correlation is always higher in bear markets and lower in bull markets for the period considered. As for Step 2 and Figures 8B and 9B, we find a weak level of dependency between the European equity market and bitcoin (EU/BTC) both in bear and bull markets; thus, the extreme correlation is reasonably stable over time. The same remarks apply for the US equity market and bitcoin (US/BTC). Similar patterns to those in Step 2 are also found in Step 3, when considering equity markets and gold, namely, either the European equity market and gold (EU/Gold) or the US equity market and gold (US/Gold). Finally, as for Step 4, we find a weak level of dependency between bitcoin and gold (BTC/Gold) over time both in bear and bull markets.

Without going further into this discussion, it seems reasonable to state that our results do remain robust over time providing early evidence of a long-lived phenomenon. We have documented a considerable body of robustness checks (model specification, validity for other equity markets and cryptocurrencies, stability over time) in support of bitcoin as a diversifier. Thus, a well-diversified equity portfolio with both bitcoin *and* gold can be the best diversification strategy in periods of stress. This evidence naturally leads to the following practical question: “How to form such portfolios using the information arising from the multivariate extreme value analysis?” We discuss these implications below.

7. Implications for asset management

In this section, we assess the practical implications of the extreme dependence structure for asset management. First, we develop a risk-return-oriented asset allocation strategy to build optimal portfolios based on tail risk constraints. Second, we present our empirical results for asset allocation and discuss the portfolio performance gains.

7.1 Optimal portfolios based on tail risk constraints

We take the point of view of investors with an equity position seeking diversification alternatives to protect their portfolio during extremely volatile periods. We study the effects of including either bitcoin or gold (and also both at the same time) in the equity position in terms of diversification gains resulting from the low extreme correlation between equity markets, on the one hand, and bitcoin or gold on the other. With our risk-return-oriented asset allocation strategy, we show that including bitcoin in an equity position provides significant gains in terms of portfolio performance. This procedure takes into account the very high level of volatility of bitcoin (although this feature is sometimes perceived as an important limitation for the use of bitcoin, as pointed out by Yermack, 2015 and Härdle et al., 2019), along with its very high price performance and its low level of correlation with traditional assets.

Since investors care about the extreme risk effects on their portfolios, we consider tail risk measures focusing on the extreme losses lying in the left tail of the distribution as a proxy of portfolio risk. This point of view is consistent with the safety-first criterion introduced by Roy (1952). This criterion is closely associated with the mean-variance rule (Levy and Sarnat, 1972), yet it explicitly takes into consideration the likelihood of large losses. Interestingly, de Haan et al. (1994) successfully improved the safety-first criterion by exploiting the heavy-tailed behavior of asset returns with the help of extreme value theory. Turning now to the role of extreme events in portfolio risk management, Longin and Solnik (2001), and Forbes and Rigobon (2002) studied the importance of the independence of extremes in portfolio design. They found that traditional dependence measures (e.g., the Pearson correlation) could easily lead to inaccurate portfolio risk assessment. Furthermore, Poon et al. (2004) showed that the failure to identify asymptotic independence - characterized in practice by very low values of extreme correlation (see subsection 3.2) - leads to overestimating the portfolio risk due to an underestimation of tail diversification gains. Considering extremes in the portfolio design is far from being a trivial exercise, as a result of the high complexity observed in their dependence structure. This high complexity mainly comes from the following aspects: (i) the presence of heavy tails in the distribution of asset returns (univariate aspect) and (ii) the tail dependence structure with different behaviors ranging from asymptotic independence to total dependence (multivariate aspect). To this end, we employ two risk measures widely used in asset and portfolio risk management, namely, the Value at Risk (VaR) and the Expected Shortfall (ES). The VaR corresponds to a loss of the position occurring with a given probability over a given time period, while the ES is the average conditional loss exceeding the VaR.[9] While traditional risk measures, such as the standard deviation of returns, work well for ordinary market conditions, tail risk measures are better fitted for extraordinary conditions (see Longin, 2000; Bali, 2000, among others).

More specifically, our risk-return-oriented allocation approach can be implemented in the following steps: (i) We first compute the tail risk measures (VaR or ES) for a given probability level for an equity position (in the European or US equity market). (ii) We then incorporate an alternative asset (bitcoin or gold) and compute its optimal weight such that the tail risk value of the new position is *equal* to the one of the initial equity position.[10] That is, we keep the same level of risk intrinsic to the initial position that has already been accepted by investors. (iii) Finally, under the same level of risk, the expected return of the portfolio is maximized. The details of our procedure to construct equal-VaR and equal-ES portfolios are given in Appendix 5. Compared to the standard portfolio optimization approach, which can lead to concentrated portfolios due to estimation errors (Merton, 1980; Jorion, 1985; Simaan, 1997; Kan and Zhou, 2007), an approach based on the safety-first criterion using tail risk constraints tends to produce more balanced portfolios in terms of risk exposure (Roy, 1952; Arzac and Bawa, 1977; de Haan et al. 1994).

To assess the benefits for investors, we consider the following three performance measures: the gain in expected return $E(r)$, the tail performance (TP) and the diversification performance (DP). The expected return $E(r)$ is the usual benchmark to measure the portfolio performance. Following Bollerslev et al. (2019) with respect to the concept of upside volatility (good volatility), the tail performance (TP) is measured by the ratio between the upside quantile of the distribution of the new position (right tail and positive return exceedances) and the upside quantile of the initial position. As our approach to design portfolios keeps the tail risk in the left tail constant by construction,

[9] From a practical point of view, despite being widely used in asset and risk management, the VaR does not properly measure diversification gains - a keystone feature in portfolio risk management - due to a lack of convexity, as shown by Artzner et al. (1999). On the other hand, the ES presents better theoretical properties than the VaR and never increases for more diversified portfolios. In other words, it is a coherent measure of risk reflecting the portfolio diversification benefits.

[10] Note that if the returns are normally distributed (the statistical assumption used in standard portfolio optimization models), our approach based on tail risk measures is equivalent to the approach based on the typical risk measure of standard deviation. When asset returns are normally distributed, the VaR for a given probability p is directly linked to the asset volatility σ by $VaR(p, \sigma) = \alpha(p)\sigma$. For example, when p is equal to 95%, the coefficient of proportionality α is equal to 1.64.

the performance gains then appear in the right tail and are captured by the TP ratio. In other words, consistent with the well-studied concept of incorporating higher order moments in portfolio selection (see Scott and Horvath, 1980; Conine and Tamarkin, 1981; Kane, 1982; Harvey et al., 2010, among others), this ratio quantifies the asymmetric effect in the distribution of returns. Following Poon et al. (2004), the diversification performance (DP) is measured by the ratio between the tail risk value (VaR or ES) of the new position with the empirical extreme correlation and the theoretical tail risk value, assuming the total dependence of negative extremes.[11] The DP ratio captures the diversification effect resulting from low levels of extreme correlation for negative return exceedances.

7.2 Empirical results for asset allocation

We now present our main empirical results for asset allocation. We consider the following different cases: two-asset portfolios (i.e., equity markets including either bitcoin or gold) and three-asset portfolios (i.e., equity markets including both bitcoin and gold).

Two-asset portfolios

Table 7 reports the optimal asset allocation based on tail risk measures. We follow our four-step procedure to present the results for the two-asset portfolios. We start with international equity markets and then include either bitcoin or gold in initial positions. We also assess the joint potential of bitcoin and gold as diversifiers. Panel A reports the optimal asset allocation for equal-VaR portfolios, and Panel B reports the optimal asset allocation for equal-ES portfolios. We consider different risk levels corresponding to the probability levels equal to 95%, 99% and 99.9% for VaR and ES.

Step 1 refers to the optimal asset allocation between the European and US equity markets (EU/US). In Panel A, for a 95%-VaR of 3.25% (obtained for an initial position in the US equity market), the optimal weights of the portfolios invested in the European and US equity markets are (30, 70). For this position, the expected return $E(r)$ is equal to 0.17% (weekly unit), the TP ratio is equal to 0.012 and the DP ratio is equal to 0.031. As we increase the risk level of the portfolio (99%-VaR of 5.74% and 99.9%-VaR of

[11] As for the case of total dependence, we simulate from a bivariate logistic distribution with parameters equal to the estimated parameters of returns presented in Tables 1-4.

9.20%), the weights of the optimal portfolio invested in the EU and US equity markets now become (58, 42) and (28, 72). In Panel B, for a 95%-ES of 4.73% (obtained for an initial position in the US equity market), the weights of the optimal portfolio invested in the EU and US equity markets are (42, 58). For this position, the expected return $E(r)$ is equal to 0.16%, the TP ratio is equal to 0.012 and the DP ratio is equal to 0.038. As we increase the risk level of the portfolio (99%-ES of 7.13% and 99.9%-ES of 10.68%), the weights of the optimal portfolio invested in the EU and US equity markets become (48, 52) and (28, 72). Note that in Step 1, the gains after combining the two equity markets are rather limited due to the high extreme correlation.

Step 2 refers to the optimal asset allocation between equity markets and bitcoin, i.e., the European equity market and bitcoin (EU/BTC) and the US equity market and bitcoin (US/BTC). Regarding the EU/BTC, for a 95%-VaR of 3.52%, the weights of the optimal portfolio invested in the European equity market and bitcoin are (92, 8). The expected return on the new portfolio $E(r)$ is equal to 0.21% (a significant increase compared to Step 1 despite the small weight of bitcoin in the new position), the TP ratio is equal to 0.117 and the DP ratio is equal to 0.389 (much higher values than those in Step 1). As we increase the risk level of the portfolio (99%-VaR of 5.95% and 99.9%-VaR of 8.80%), the weights of the optimal portfolio invested in the European equity market and bitcoin become (89, 11) and (85, 15). The increase in the bitcoin weights is consistent with the decrease in extreme correlation documented in Section 4. The gains in terms of portfolio performance also increase. Even though bitcoin exhibits an extreme volatile price behavior (sometimes using bitcoin is perceived as an important limitation, as we already mentioned), our risk-return-oriented allocation approach shows that bitcoin can be optimally introduced in an equity position under the tail risk constraint. The same remarks apply for the US equity market and bitcoin (US/BTC).

Step 3 refers to the optimal asset allocation between equity markets and gold, i.e., the European equity market and gold (EU/Gold) and the US equity market and gold (US/Gold). Regarding the EU/Gold, for 95%-VaR of 3.48%, the weights of the optimal portfolio invested in the EU and gold are (95, 5). The expected return on the new portfolio $E(r)$ is equal to 0.10% (a lower increase compared to Step 1 due to the low weight of gold and its low expected return observed over the period), the TP ratio is equal to 0.051 and the DP ratio is equal to 0.060 (higher values than those in Step 1). As we increase the risk level of the portfolio (99%-VaR of 5.81% and 99.9%-VaR of 9.32%), the weights of the optimal portfolio invested in the EU and gold become (97, 3) and (95, 5). Although the gains are somehow limited, higher weights for gold can reduce the tail risk exposure of an investor, as documented in subsection 4.3. Compared to Step 2, the inclusion of gold has a lower impact on the portfolio performance than the inclusion of bitcoin. The same remarks apply for the US equity market and gold (US/Gold), with the only differences being observed in VaR and ES, which are always lower than the European equity market and gold (EU/Gold).

Step 4 refers to the optimal asset allocation between bitcoin and gold (BTC/Gold). For a 95%-VaR of 3.20% (obtained for an initial position in gold), the weights of the optimal portfolio invested in bitcoin and gold are (4, 96). The expected return on the new portfolio $E(r)$ is equal to 0.03%, the TP ratio is equal to 0.203 and the DP ratio is equal to 0.219. As we increase the risk level of the portfolio (99%-VaR of 5.08% and 99.9%-VaR of 7.80%), the weights of the optimal portfolio invested in bitcoin and gold become (10, 90) and (16, 84). We observe that as the level of risk increased, the position in bitcoin is also increased. This is explained by the low extreme correlation between bitcoin and gold as we go further in distribution tails. This implies that bitcoin and gold cannot only be used together as diversifiers when extreme price movements occur in equity markets, but also a position in both of them achieves additional performance gains under our risk-return-oriented allocation strategy (as confirmed for the three-asset portfolios studied in the following subsection).

Summarizing the case of the two-asset portfolios, the introduction of bitcoin in an equity position achieves high performance for equal levels of tail risk measured by either the VaR or the ES. In spite of the fact that a position in bitcoin per se can be extremely risky, its certain features (e.g. high price performance along with low levels of extreme correlation with traditional assets) can be considered useful for practitioners under an appropriate asset allocation strategy. Including bitcoin in an equity position cannot only provide a risk-reduction benefit but also increase substantially the portfolio performance by keeping the portfolio risk constant. The introduction of gold also achieves some performance, albeit lower than that of bitcoin.

Three-asset portfolios

We now turn our attention to the assessment of the bitcoin-and-gold potential as joint diversifiers for equity positions. We study the portfolio performance of an equity position including both bitcoin and gold: the European equity market, bitcoin and gold

(EU/BTC/Gold) and the US equity market, bitcoin and gold (US/BTC/Gold). Our findings fully support the use of both assets in a portfolio due to their low levels of extreme correlation (Step 4).

Table 8 reports the optimal asset allocation based on tail risk measures for the three-asset portfolios. We start with international equity markets as initial positions and then include both bitcoin and gold to assess their joint potential as diversifiers. Panel A reports the optimal asset allocation for equal-VaR portfolios, and Panel B reports the optimal asset allocation for equal-ES portfolios. For comparison purposes, we consider again different risk levels, i.e., 95%, 99% and 99.9% for both VaR and ES. More specifically, in Panel A, under our risk-return-oriented asset allocation strategy, for a 95%-VaR of 3.52% (obtained for an initial position in the EU equity market), the optimal weights of the portfolios invested in the EU equity market, bitcoin and gold are (64, 15, 21). The optimal weight of bitcoin in the European equity position is now almost doubled compared to the case of two-asset portfolios and Step 2. The optimal contribution of gold is also increased. For this position, the expected return $E(r)$ is equal to 0.29%, while the TP and DP ratios are equal to 0.247 and 0.384, respectively. As we increase the risk level of the portfolio (99%-VaR of 5.92% and 99.9%-VaR of 8.86%), the weights of the optimal portfolio now become (55, 17, 28) and (59, 19, 22). In Panel B, for a 95%-ES of 4.77% (obtained for an initial position in the EU equity market too), the weights are now (62, 15, 23). For this position, the gains are: 0.29% for expected return $E(r)$, 0.253 for the TP ratio and for the DP ratio they are equal to 0.395. As we increase the risk level of the portfolio (99%-ES of 7.13% and 99.9%-ES of 10.68%), the optimal weights become (61, 20, 19) and (53, 25, 22). Similar remarks apply to the US equity market, bitcoin and gold (US/BTC/Gold), although the increase in performance is higher than for the European equity market. In consistence with the concept of diversification in diversifiers (D-*in*-D), the use of two diversifiers with low levels of extreme correlation leads to an increase in their weights and an increase in portfolio performance.

To clarify the previous point, let us consider the following simple interpretation where two opposite effects take place - the diversification effect and the volatility effect - both affecting portfolio risk and portfolio performance. The diversification effect tends to reduce the portfolio risk, as assets with low levels of extreme correlation are now added in the portfolio. Hyung and De Vries (2005) noted a similar effect when more assets were to be included. The volatility effect tends to increase portfolio performance, as assets with higher levels of volatility are now included in the portfolio. Under our risk-return-oriented allocation strategy, in order to keep the portfolio risk constant, a detour via increasing the weights of riskier assets (e.g. bitcoin) is necessary. If this is not the case, lower risk levels will arise due to the diversification effect. However, increasing the weights of riskier assets presenting higher levels of volatility (both downside and upside) will result in a positive effect on portfolio performance (volatility effect). In the case of asymmetry in return distribution, the volatility effect is higher.

Summarizing the case of the three-asset portfolios, we confirm the most salient result in terms of portfolio risk management obtained in Section 5 about the low extreme correlation between bitcoin and gold. The portfolio analysis shows that the concept of D-*in*-D is in practice beneficial, indeed. In particular, the inclusion of both diversifiers significantly improves the performance with regard to portfolio gains under tail risk constraints (compared to the case of two-asset portfolios). Thus, using both bitcoin *and* gold is the best answer to diversify an equity position in times of extreme price fluctuations. This is our recommendation for asset managers.

8. Conclusion

Is bitcoin the new digital gold? This question has lately begun to concern practitioners and the financial press. In order to provide a rigorous answer to this question, we investigate the potential benefits of bitcoin during extremely volatile periods in financial markets. This is a challenging exercise, and one should exert great care to avoid misleading results, mainly due to the high complexity observed in the dependence structure of extreme returns. In this study, to avoid wrong conclusions, we use the extreme value theory, which is the proper statistical approach to study this issue. In a multivariate framework, by focusing on the correlation of return exceedances, we analyze the tail dependence structure of international equity markets, i.e., Europe and the United States vis-à-vis bitcoin and gold, in a pairwise comparison.

We develop a research strategy in four steps. In Step 1, we consider a position in equity markets and find - similarly to previous studies - that the correlation of extreme returns increases during stock market crashes and decreases during stock market booms. In Step 2, we combine each equity position with bitcoin and find that the correlation of extreme returns decreases sharply during both market booms and crashes. This indicates that bitcoin can play an important role in asset management by providing diversification benefits in periods of extreme price volatility. In Step 3, we combine each equity position with gold and obtain a similar result. This confirms its well-recognized status as a safe haven in asset management when a crisis occurs. In Step 4, we combine bitcoin and gold and find a low correlation of return exceedances. This indicates that both assets can be useful together in times of turbulence in financial markets. Further analysis also suggests that bitcoin can be useful in asset management in practice, as the introduction of bitcoin along with gold substantially improves the risk-return characteristics of equity positions with tail risk constraints. Such evidence shows that bitcoin can be considered the new digital gold, yet gold itself can still play an important role in portfolio risk management.

Overall, from a portfolio risk management point of view, concerning the question "bitcoin or gold", having both bitcoin and gold as complementary diversifiers is the best answer to diversify a position in equity markets when extreme price movements take place. To conclude, the diversification in diversifiers (D-*in*-D) is the optimal strategy in terms of bringing about additional value for investors.

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Appendix 1. Derivation of the maximum likelihood function of the logistic model

To estimate the parameters of the model presented in Section 3, we use the maximum likelihood method, which was developed by Ledford and Tawn (1997) based on the threshold-censored likelihood method of Smith et al. (1997). The method also applied by Longin and Solnik (2001) is reproduced below. This appendix presents the construction of the likelihood function in detail.

The method is based on a set of assumptions. The observations are assumed to be independent. The thresholds u_1 and u_2 are used to select the exceedances (or, equivalently, the corresponding tail probabilities p_1 and p_2) and they are independent of the variables and time. The method is also based on a censoring assumption. For thresholds u_1 and u_2, the space of the events is divided into four regions given by $\{A_{jk}; j = I(X_1 > u_1), k = I(X_2 > u_2)\}$, where $I(\cdot)$ is the indicator function. The method treats observations below the threshold as censored data, and thus, no assumption is made for the dependence structure outside A_{11}. Finally, the dependence in extremes is modeled by using a logistic function denoted by D_X.

The likelihood contribution corresponding to the observation of (X_{1t}, X_{2t}) at time t falling in region A_{jk} is denoted by $L_{jk}(X_{1t}, X_{2t})$ and given by the following:

$$\begin{aligned}
L_{00}(X_{1t}, X_{2t}) &= exp\big(-D_X(Y_1, Y_2)\big) \\
L_{10}(X_{1t}, X_{2t}) &= \frac{\partial F_X^u(X_{1t}, X_{2t})}{\partial X_{1t}} \\
&= exp\big(-D_X(Z_1, Y_2)\big)\frac{\partial D_X}{\partial X_{1t}}(Z_1, Y_2)K_1 \\
L_{01}(X_{1t}, X_{2t}) &= \frac{\partial F_X^u(X_{1t}, X_{2t})}{\partial X_{2t}} \\
&= exp\big(-D_X(Y_1, Z_2)\big)\frac{\partial D_X}{\partial X_{2t}}(Y_1, Z_2)K_2 \qquad \text{(A1.1)} \\
L_{11}(X_{1t}, X_{2t}) &= \frac{\partial^2 F_X^u(X_{1t}, X_{2t})}{\partial X_{1t}\partial X_{2t}} \\
&= exp\big(-D_X(Z_1, Z_2)\big)\left(\frac{\partial D_X}{\partial X_{1t}}(Z_1, Z_2)\frac{\partial D_X}{\partial X_{2t}}(Z_1, Z_2)\right. \\
&\left. - \frac{\partial^2 D_X}{\partial X_{1t}\partial X_{2t}}(Z_1, Z_2)\right)K_1K_2
\end{aligned}$$

where the variables Y_i, Z_i and K_i for $i = 1{,}2$ are defined by the following:

$$Y_i = -1/\log F_{X_i}^{u_i}\,(u_i)$$

$$Z_i = -1/\log F_{X_i}^{u_i}(X_{it}) \tag{A1.2}$$

$$K_i = -p_i\sigma_i^{-1}\,(1+\xi_i(X_{it}-u_i)/\sigma_i)_+^{-(1+\xi_i)/\xi_i} Z_i^2\, exp(1/Z_i)$$

The likelihood contribution from the observation of (X_{1t}, X_{2t}) at time t for the bivariate distribution of exceedances described by a set of parameters $\Phi = (p_1, p_2, \sigma_1, \sigma_2, \xi_1, \xi_2, \alpha)$ is given by the following:

$$L(X_{1t}, X_{2t}, \Phi) = \sum_{j,k\in\{0,1\}} L_{jk}(X_{1t}, X_{2t}) I_{jk}(X_{1t}, X_{2t}) \tag{A1.3}$$

where $I_{jk}(X_{1t}, X_{2t}) = I\{(X_{1t}, X_{2t}) \in A_{jk}\}$. Hence, the likelihood for a set of T independent observations is given by the following:

$$L(\{X_{1t}, X_{2t}\}_{t=1,T}, \Phi) = \prod_{t=1}^{T} L(X_{1t}, X_{2t}, \Phi) \tag{A1.4}$$

Appendix 2. Procedure to obtain stationary return series

To apply the extreme value theory, it is important to work with stationary time series. To deal with this issue, we build on the procedure developed by Gallant et al. (1992) to remove trends and the work of McNeil and Frey (2000) to take into account heteroskedasticity due to volatility clusters.[12] We present the data adjustment procedure used in this study in detail below.

Step 1: Detrending the mean

First, we detrend the mean by regressing the raw original series on a set of explanatory variables that take into account the time trends (linear and quadratic) and several seasonality effects, as follows:

$$\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{\beta} + \boldsymbol{u} \tag{A2.1}$$

with $\boldsymbol{y}$ being log-returns. The matrix $\boldsymbol{x}$ comprises the following regressors: a constant term, dummy variables to take into account monthly effects: one dummy variable for each month of the year except December to avoid multicollinearity, and two variables to take into account time trends (a linear and a quadratic one). In total, $\boldsymbol{x}$ comprises 13 regressors including the constant. The aforementioned regressors are meant to take into account the seasonality of a return series.

Step 2: Detrending the variance

Second, we detrend the variance of the time series by running the subsequent regression, as follows:

$$\mathbf{log}\,\boldsymbol{u}^{\mathbf{2}} = \boldsymbol{x}'\boldsymbol{\gamma} + \boldsymbol{\epsilon} \tag{A2.2}$$

where it has to be noted that the same set of explanatory variables is used to remove the trend from the variance.

Step 3: Adjusting the time series

Third, we perform the following transformation to compute the adjusted time series:

[12] The application of extreme value theory on real data is based on the assumption of independent consecutive extreme values. However, with volatility clustering, this assumption is violated. This leads to model misspecification, especially in terms of the degree of heaviness of distribution tails (see Kearns and Pagan, 1997; Chavez-Demoulin and Davison, 2012, Massacci, 2017, among others).

$$\boldsymbol{y}_{\boldsymbol{adj}} = \boldsymbol{a} + \boldsymbol{b}\left(\frac{\widehat{\boldsymbol{u}}}{\frac{e^{x'\gamma}}{2}}\right) \tag{A2.3}$$

where coefficients $\boldsymbol{a}$ and $\boldsymbol{b}$ in Equation (A2.3) are determined by solving a system of two equations with two unknowns, where the adjusted time series is required to have the same mean and variance as the original series.

Step 4: GARCH-type filters

Fourth, we fit GARCH processes with various forms of marginal distribution (to model the alternance of low and high volatility periods). We use different combinations for processes' values ranging from zero lag to a maximum of three lags. We select the best model based on the Akaike Information Criterion (AIC). By fitting such a process, the sequence of residuals is based on the assumption of independent and identically distributed random variables.

Appendix 3. Data visualization with nonparametric copulas

We use nonparametric copulas to conduct a preliminary analysis of the dependence patterns in our data. Nonparametric copulas are flexible, as they directly fit the data. They can be used to assess the validity of a parametric model (the logistic model with the Gumbel-Hougaard copula in our case) by taking into consideration the misspecification risk (a type of model risk).

In this appendix, we present the statistical procedure proposed by Geenens et al. (2017) via a kernel-type copula density estimator. We first provide a brief discussion of this statistical procedure. Then, following our four-step research strategy, we provide a brief representation of the kernel-type copula density estimation by using surface plots. Surface plots allow us to visually determine the functional relationships of the data to obtain preliminary evidence of tail dependence patterns.

Deheuvels (1978) proposed an earlier nonparametric estimate of the copula function $C(y_1, y_2)$, known as empirical copula (where $C(y_1, y_2) = Pr(Y_1 \leq y_1, Y_2 \leq y_2) = F\left(F_{X_1}^{-1}(y_1), F_{X_2}^{-1}(y_2)\right)$ where $y_1, y_2 \in [0,1]$, see subsection 3.2). The empirical copula is essentially the empirical distribution of the rank-transformed data. A direct derivation of the empirical copula is the estimation of the empirical copula density, which can be considered as a nonparametric approach. The empirical copula density, however, is heavily affected by boundary bias issues (i.e., the empirical copula density is not a consistent estimator on the boundaries of the interval [0,1]). The so-called boundary bias is present at the boundaries as well as in their neighborhood, as noted by Charpentier et al. (2007). The use of kernel methods was an earlier solution to these issues. Behnen et al. (1985), and Gijbels and Mielniczuk (1990) proposed procedures relying on symmetric kernels. Although kernel estimators are considered to be more suitable nonparametric density estimators, they do not eliminate the boundary bias. More recently, Geenens et al. (2017) addressed this problem by transforming the uniform marginals of the copula density into normal distributions via the probit function. They proposed an estimate of the copula density via back-transformation by using a local likelihood estimator with nearest-neighbor bandwidths, which is accomplished without boundary problems (we refer to the studies of Geenens et al., 2017 and Nagler, 2018, for more information regarding the estimation procedure of the nonparametric density).

Figure 10 depicts surface plots for the nonparametric kernel-type copula density estimator for international equity markets, including bitcoin or gold. Figure 10A refers to equity markets (Step 1), i.e., European and US equity markets. Figures 10B and 10C refer to equity markets and bitcoin (Step 2), i.e., the European equity market and bitcoin and the US equity market and bitcoin, respectively. Figures 10D and 10E refer to equity markets and gold (Step 3), i.e., the European equity market and gold and the US equity market and gold, respectively. Figure 10F refers to bitcoin and gold (Step 4). More specifically, regarding Step 1 and Figure 10A, we find that the density is significantly higher in bear markets and lower in bull markets. This provides initial graphical evidence of strong tail dependence between European and US equity markets during stock market crashes. Regarding Step 2 and Figures 10B and 10C, we find a weak level of dependency between equity markets and bitcoin, both in bear and bull markets. Regarding Step 3 and Figures 10D and 10E, a similar result is obtained as in the previous step. Finally, regarding Step 4 and Figure 10F, we also find a weak level of dependency, both in bear and bull markets.

Appendix 4. Computation of optimal threshold levels

Over a high threshold u, the peaks-over-threshold method constitutes an efficient way to model extremes via the general Pareto distribution (GPD). However, one of the most important factors when dealing with extremes is the selection of threshold u. A too low threshold value induces a significant estimation bias due to observations not belonging to the distribution tails considered as exceedances (with the asymptotic assumption being violated). A too high threshold value leads to inefficiency with increasing standard errors due to the reduced size of the estimation sample (although the asymptotic assumption is stronger and leads to bias reduction). An optimal threshold optimizes the trade-off between inefficiency and sample bias.

In the existing literature, several approaches to this issue have been proposed. In this paper, we apply the procedure proposed by Gkillas et al. (2017) using the parametric bootstrap goodness-of-fit test of Villasenor-Alva and Gonzalez-Estrada (2009) for the computation of optimal thresholds. In applying this procedure, we take into consideration the error of accepting that the GPD is a distribution for a random sample, defined by a threshold u, when u is not appropriate (be it either too high or too low). We minimize this error via this powerful goodness-of-fit test. This test can provide results for the whole parameter space in relation to other goodness-of-fit tests proposed in the literature (e.g. Choulakian and Stephens, 2001 and Meintanis and Bassiakos, 2007). Furthermore, we apply a parametric bias-corrected approach based on the maximum likelihood procedure to reduce the sample bias observed in small samples (see subsection 3.2). We describe the procedure for the selection of optimal thresholds in detail in this appendix.

Let $X = \{X_1, X_2, \dots, X_n\}$ be a sequence of independent and identically distributed random variables defined on the positive real numbers with a continuous cumulative distribution function F_X, for $i = 1, 2, .., n$. Additionally, let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the corresponding order statistics. Our approach is developed in the following steps.

Step 1: Extracting subsequences associated with unknown thresholds

We extract n subsequences from X, such that $X'_{k_n} = \left\{X'_{(k_n)}, X'_{(k_n+1)}, \dots, X'_{(n)}\right\}^{k_n} \subseteq X$ for $k_n = 1, \dots, n$, where k_n corresponds to a number of upper order statistics and can be associated with the unknown threshold u of the GPD defined in subsection 3.1.

Step 2: Extracting the optimal solution

We apply an iterative n-step algorithm and we select the k_n number of upper order statistic that corresponds to the maximal p-value (p) of the intersection-union goodness-of-fit test of Villasenor-Alva and Gonzalez-Estrada (2009), as follows:

$$u = X'_{(k_n)} \mathrel{\widehat{=}} \max_{k_n=1,\dots,n}\{p_{(k_n)}, p_{(k_n+1)}, \dots, p_{(n)}\},\ k_n \in \{1, \dots, n\} \qquad \text{(A4.1)}$$

for the null hypothesis, i.e., $H_0: F_X^u(x) \sim G_{\xi,\sigma}(x)$, defined by two subclasses of GPD, the A^+, which corresponds to $H_0^+: F_X^u(x) \sim G_{\xi,\sigma}(x)$ with $\xi \geq 0$, and the A^-, which corresponds to $H_0^-: F_X^u(x) \sim G_{\xi,\sigma}(x)$ with $\xi < 0$. Thus, $H_0: F \in (A^+ \cup A^-)$, which is rejected whenever both hypotheses H_0^+ and H_0^- are rejected.

Step 3: Extracting the optimal threshold

The optimal threshold u corresponds to the optimal k_n^{th} upper order statistic of the previous step.

Step 4: Bootstrapping the goodness-of-fit test

We apply this procedure in each distribution tail separately for 999 bootstrap samples.

Appendix 5. Risk-return portfolio optimization

We propose a risk-return-oriented optimal asset allocation strategy to take the point of view of investors who care about the extreme losses on their portfolios. In this case, their risk preferences can be expressed in terms of tail risk (Jansen et al., 2000). We study the effects of including a riskier asset in an initial position in terms of diversification gains to highlight the importance of potential low extreme correlation of asset returns considered. The details of the portfolio optimization procedure, namely, the equal-VaR or equal-ES portfolios, are presented below.

Consider a finite number of risky assets $i = 1, 2, \dots, N$. For a given time period, these assets produce returns $X = \{X_1, X_2, \dots, X_N\}$. The returns measure the relative change, either an increase or a decrease of the asset prices during the time period examined. The returns are unknown at the time of portfolio allocation, and hence, they are viewed as random variables. Let us suppose that an investor has a budget of one unit - without loss of generality - and can select on the positions $w = \{w_1, w_2, \dots, w_N\}$ in these assets, such that $w_i \geq 0$, that is, no short sales are allowed, and $\sum_{i=1}^{N} w_i = w^T 1_N = 1$, which reflects the budget constraint. Consequently, the return of the portfolio for a given time period is equal to $r = \sum_{i=1}^{N} w_i X_i = w^T X$, while the expected return is equal to $E(r) = w^T E(X)$.

Let $TR(p) = \{TR_1(p), TR_2(p), \dots, TR_N(p)\}$ be a tail risk measure (e.g. the VaR or ES) with the corresponding order statistics $TR_{(1)}(p) \leq TR_{(2)}(p) \leq \cdots \leq TR_{(N)}(p)$. For a level of risk being equal to $TR_{(1)}(p)$, let us consider the solution of the following optimization problem:

$$\begin{aligned} \max_{w \in \mathbb{R}^N} \;& E(r) \stackrel{\text{def}}{=} w^T E(X) \\ s.t. \;& TR(p, w^T X) := TR_{(1)}(p), \\ & w^T 1_N = 1, \\ & w \geq 0 \end{aligned} \tag{A5.1}$$

where p corresponds to the p-quantile of the cumulative distribution, defining the loss to be expected in $(p \cdot 100)\%$ of the times.

The optimization problem above is based on tail risk constraints, and thus, it is considered inherently more difficult than the variance optimization problem, for example. An optimization problem based on tail risk measures, such as the VaR which is nonconvex (see Artzner et al., 1999), can have several local maxima increasing the computational complexity. In this appendix, we apply an iterative algorithm to solve the problem of Equation (A5.1). In particular, the optimization procedure is implemented in the following three steps.

Step 1: Computation of tail risk measures

First, we compute the tail risk measures (VaR or ES) for a given probability level p for the position $TR_{(1)}(p)$ corresponding to the lowest level of tail risk of the assets considered. The risk measures are estimated as functions of the parameters of the general Pareto distribution (GPD) defined in subsection 3.1 by the following:

$$VaR_{(1)}(p) = u_{(1)} + \frac{\sigma_{(1)}}{\xi_{(1)}}\left(\frac{n}{k_{n(1)}}(1-p)^{-\xi_{(1)}} - 1\right) \tag{A5.2}$$

$$ES_{(1)}(p) = \frac{1}{1-\xi_{(1)}}\left(VaR_{(1)}(p) + \sigma_{(1)} - u_{(1)}\xi_{(1)}\right) \tag{A5.3}$$

where $u_{(1)}$ represents the threshold, $k_{n(1)}$ is the number of exceedances over threshold $u_{(1)}$, n is the number of observations, $\sigma_{(1)} > 0$ is the scale parameter, and $\xi_{(1)} \in \mathbb{R}$ corresponds to the tail index.

Step 2: Computation of local solutions

Second, we incorporate $N-1$ riskier assets (in terms of tail risk) and compute their optimal weights, such that the tail risk value of the new position is equal to the tail risk value of the position given by $TR_{(1)}(p)$. However, due to the high complexity observed in the dependence structure of extremes, an exact solution is not always obtained for the first constraint $TR(p, w^T X) := TR_{(1)}(p)$ as the events become more extreme and the dependence structure tends to total dependence. In this case, an approximate solution is obtained by the following:

$$\arg\min_{w\in\mathbb{R}^N}\left|TR_{(1)}(p) - TR(p, w^T X)\right| \tag{A5.4}$$

Step 3: Selection of the optimal solution

Third, among the local solutions obtained in Step 2 satisfying the tail risk constraint, we select the optimal solution which maximizes the expected return $E(r)$ of the portfolio, that is:

$$\max_{w\in\mathbb{R}^N} E(r) \stackrel{\text{def}}{=} w^T E(X) \tag{A5.5}$$

where $E(r)$ represents the expected return of the portfolio, the usual variable maximized in portfolio analysis.

Table 1. Estimation of the bivariate distribution of the return exceedances for the European and US equity markets

Panel A: Negative return exceedances

	Parameters of the model								Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{US}	σ^{US}	ξ^{US}	α	ρ	$H_0\colon \rho = 0$	$H_0\colon \rho = \rho_{nor}^{f.s}(u)$	$H_0\colon \rho = 1$
5%	0.036	0.015	0.057	0.031	0.026	-0.505	0.379	0.890	203.487	3.726	25.008
		(0.005)	(0.280)		(0.008)	(0.278)	(0.058)	(0.004)	[0.000]	[0.000]	[0.000]
10%	0.029	0.010	0.267	0.023	0.013	0.058	0.408	0.841	68.944	4.124	13.057
		(0.003)	(0.237)		(0.004)	(0.249)	(0.040)	(0.012)	[0.000]	[0.000]	[0.000]
20%	0.018	0.014	0.017	0.013	0.014	0.013	0.409	0.831	36.642	2.152	7.426
		(0.002)	(0.107)		(0.002)	(0.136)	(0.029)	(0.023)	[0.000]	[0.031]	[0.000]
30%	0.008	0.019	−0.099	0.007	0.015	−0.013	0.378	0.875	28.733	1.361	4.106
		(0.002)	(0.068)		(0.002)	(0.100)	(0.022)	(0.030)	[0.000]	[0.173]	[0.000]
40%	0.003	0.019	-0.087	0.003	0.014	0.009	0.365	0.877	26.008	0.839	3.638
		(0.002)	(0.064)		(0.002)	(0.088)	(0.018)	(0.034)	[0.000]	[0.400]	[0.000]
50%	0.000	0.019	-0.075	0.000	0.013	0.035	0.351	0.888	24.641	0.694	3.256
		(0.002)	(0.063)		(0.001)	(0.079)	(0.016)	(0.036)	[0.000]	[0.487]	[0.001]
3.01%	0.042	0.021	-0.130	0.041	0.023	-0.393	0.349	0.878	63.419	3.130	8.678
3.00%		(0.008)	(0.288)		(0.007)	(0.260)	(0.071)	(0.014)	[0.000]	[0.001]	[0.000]

Panel B: Positive return exceedances

	Parameters of the model								Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{US}	σ^{US}	ξ^{US}	α	ρ	$H_0\colon \rho = 0$	$H_0\colon \rho = \rho_{nor}^{f.s}(u)$	$H_0\colon \rho = 1$
50%	0.000	0.017	-0.151	0.000	0.014	-0.122	0.385	0.864	24.194	0.169	27.155
		(0.001)	(0.041)		(0.001)	(0.056)	(0.016)	(0.036)	[0.000]	[0.865]	[0.000]
40%	0.006	0.014	-0.094	0.004	0.012	-0.055	0.435	0.810	25.739	0.619	30.949
		(0.001)	(0.058)		(0.001)	(0.074)	(0.020)	(0.031)	[0.000]	[0.535]	[0.000]
30%	0.010	0.012	-0.050	0.008	0.011	-0.012	0.472	0.785	29.597	0.629	36.908
		(0.001)	(0.079)		(0.001)	(0.094)	(0.024)	(0.027)	[0.000]	[0.529]	[0.000]
20%	0.015	0.012	-0.067	0.012	0.011	-0.008	0.512	0.746	39.503	0.224	52.186
		(0.002)	(0.086)		(0.002)	(0.121)	(0.031)	(0.019)	[0.000]	[0.822]	[0.000]
10%	0.024	0.009	0.021	0.019	0.012	-0.072	0.566	0.686	153.940	0.486	223.625
		(0.002)	(0.132)		(0.003)	(0.165)	(0.043)	(0.004)	[0.000]	[0.626]	[0.000]
5%	0.031	0.008	0.094	0.027	0.012	-0.105	0.696	0.521	17.260	0.548	32.592
		(0.002)	(0.203)		(0.004)	(0.250)	(0.068)	(0.030)	[0.000]	[0.558]	[0.000]
2.00%	0.040	0.003	0.668	0.027	0.012	-0.105	0.795	0.384	6.003	1.379	15.239
5.01%		(0.002)	(0.538)		(0.004)	(0.250)	(0.087)	(0.064)	[0.000]	[0.168]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for the European and US equity markets represented by the STOXX Europe 600 index and the S&P 500 index. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{US}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0\colon \rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering the following two special cases: asymptotic independence $H_0\colon \rho = 0$ and total dependence $H_0\colon \rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 2A. Estimation of the bivariate distribution of the return exceedances for the European equity market and bitcoin

Panel A: Negative return exceedances

	Parameters of the model								Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{BTC}	σ^{BTC}	ξ^{BTC}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
5%	0.035	0.013	-0.084	0.193	0.073	-0.182	0.999	0.019	0.373	0.273	19.616
		(0.008)	(0.549)		(0.045)	(0.555)	(0.043)	(0.050)	[0.709]	[0.784]	[0.000]
10%	0.029	0.011	0.026	0.149	0.077	-0.176	0.923	0.170	20.099	1.988	97.826
		(0.003)	(0.246)		(0.022)	(0.219)	(0.040)	(0.008)	[0.000]	[0.047]	[0.000]
20%	0.016	0.019	-0.233	0.075	0.112	-0.288	0.868	0.234	23.400	0.426	76.600
		(0.003)	(0.121)		(0.020)	(0.124)	(0.034)	(0.010)	[0.000]	[0.795]	[0.000]
30%	0.007	0.022	-0.255	0.027	0.131	-0.312	0.801	0.364	33.791	0.343	59.062
		(0.003)	(0.096)		(0.018)	(0.092)	(0.030)	(0.011)	[0.000]	[0.732]	[0.000]
40%	0.002	0.021	-0.216	0.011	0.104	-0.166	0.735	0.472	25.625	0.417	28.636
		(0.003)	(0.087)		(0.015)	(0.100)	(0.026)	(0.018)	[0.000]	[0.677]	[0.000]
50%	0.000	0.021	-0.193	0.000	0.092	-0.086	0.715	0.477	21.525	2.316	23.640
		(0.003)	(0.086)		(0.013)	(0.106)	(0.023)	(0.022)	[0.000]	[0.021]	[0.000]
11.11%	0.028	0.009	0.136	0.160	0.063	-0.061	0.924	0.164	20.456	0.943	104.412
9.20%		(0.003)	(0.252)		(0.021)	(0.253)	(0.039)	(0.008)	[0.000]	[0.346]	[0.000]

Panel B: Positive return exceedances

	Parameters of the model								Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{BTC}	σ^{BTC}	ξ^{BTC}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
50%	0.000	0.018	-0.295	0.000	0.094	-0.042	0.649	0.609	21.568	1.083	34.800
		(0.002)	(0.079)		(0.012)	(0.088)	(0.019)	(0.028)	[0.000]	[0.279]	[0.000]
40%	0.006	0.015	-0.242	0.024	0.099	-0.078	0.749	0.456	25.615	0.029	55.721
		(0.002)	(0.109)		(0.013)	(0.095)	(0.026)	(0.018)	[0.000]	[0.977]	[0.000]
30%	0.010	0.013	-0.200	0.050	0.099	-0.088	0.810	0.353	31.713	0.001	89.500
		(0.002)	(0.149)		(0.016)	(0.111)	(0.030)	(0.011)	[0.000]	[1.000]	[0.000]
20%	0.014	0.016	-0.397	0.089	0.095	-0.084	0.868	0.265	105.568	0.555	398.491
		(0.002)	(0.126)		(0.019)	(0.142)	(0.034)	(0.003)	[0.000]	[0.579]	[0.000]
10%	0.025	0.012	-0.437	0.155	0.075	0.051	0.898	0.209	15.893	2.771	75.683
		(0.000)	(0.077)		(0.025)	(0.274)	(0.044)	(0.013)	[0.000]	[0.006]	[0.000]
5%	0.031	0.013	-0.716	0.202	0.139	-0.470	0.949	0.084	3.896	0.686	46.574
		(0.000)	(0.060)		(0.081)	(0.530)	(0.047)	(0.021)	[0.000]	[0.492]	[0.000]
11.11%	0.023	0.013	-0.465	0.114	0.080	0.004	0.906	0.175	65.110	0.113	371.015
11.14%		(0.019)	(0.065)		(0.024)	(0.239)	(0.065)	(0.003)	[0.000]	[0.910]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for the European equity market, represented by the STOXX Europe 600 index, and bitcoin. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{BTC}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering the following two special cases: asymptotic independence $H_0: \rho = 0$ and total dependence $H_0: \rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 2B. Estimation of the bivariate distribution of the return exceedances for the US equity market and bitcoin

Panel A: Negative return exceedances

Parameters of the model									Wald tests		
p	u^{US}	σ^{US}	ξ^{US}	u^{BTC}	σ^{BTC}	ξ^{BTC}	α	ρ	$H_0{:}\,\rho=0$	$H_0{:}\,\rho=\rho_{nor}^{f.s}(u)$	$H_0{:}\,\rho=1$
5%	0.031	0.026	-0.505	0.193	0.073	-0.182	0.936	0.123	10.250	2.987	73.083
		(0.008)	(0.278)		(0.044)	(0.554)	(0.042)	(0.012)	[0.000]	[0.002]	[0.000]
10%	0.019	0.019	-0.237	0.149	0.077	-0.176	0.919	0.186	17.183	0.368	75.152
		(0.006)	(0.259)		(0.022)	(0.219)	(0.041)	(0.011)	[0.000]	[0.713]	[0.000]
20%	0.012	0.010	0.172	0.075	0.112	-0.288	0.827	0.333	444.804	2.343	890.161
		(0.002)	(0.207)		(0.020)	(0.124)	(0.036)	(0.001)	[0.000]	[0.019]	[0.000]
30%	0.006	0.013	0.011	0.027	0.131	-0.312	0.771	0.414	37.986	1.443	53.672
		(0.002)	(0.125)		(0.018)	(0.092)	(0.031)	(0.011)	[0.000]	[0.149]	[0.000]
40%	0.002	0.013	0.030	0.011	0.104	-0.166	0.720	0.499	27.258	0.376	27.311
		(0.002)	(0.110)		(0.015)	(0.100)	(0.026)	(0.018)	[0.000]	[0.707]	[0.000]
50%	0.000	0.012	0.036	0.000	0.092	-0.086	0.686	0.547	23.502	0.761	19.614
		(0.002)	(0.099)		(0.01)3	(0.106)	(0.023)	(0.023)	[0.000]	[0.446]	[0.000]
7.28%	0.024	0.018	-0.248	0.160	0.063	-0.061	0.904	0.192	9.803	1.215	41.341
9.20%		(0.008)	(0.397)		(0.021)	(0.253)	(0.048)	(0.020)	[0.020]	[0.225]	[0.000]

Panel B: Positive return exceedances

Parameters of the model									Wald tests		
p	u^{US}	σ^{US}	ξ^{US}	u^{BTC}	σ^{BTC}	ξ^{BTC}	α	ρ	$H_0{:}\,\rho=0$	$H_0{:}\,\rho=\rho_{nor}^{f.s}(u)$	$H_0{:}\,\rho=1$
50%	0.000	0.015	-0.322	0.000	0.094	-0.042	0.657	0.599	22.639	0.442	37.187
		(0.002)	(0.075)		(0.012)	(0.088)	(0.021)	(0.026)	[0.000]	[0.658]	[0.000]
40%	0.005	0.012	-0.252	0.024	0.099	-0.078	0.710	0.514	27.222	0.751	52.489
		(0.002)	(0.102)		(0.013)	(0.095)	(0.026)	(0.019)	[0.000]	[0.453]	[0.000]
30%	0.008	0.012	-0.285	0.050	0.099	-0.088	0.835	0.312	30.213	2.012	96.607
		(0.002)	(0.120)		(0.016)	(0.111)	(0.029)	(0.010)	[0.000]	[0.044]	[0.000]
20%	0.012	0.012	-0.326	0.089	0.095	-0.084	0.851	0.293	249.830	0.389	853.126
		(0.002)	(0.156)		(0.019)	(0.142)	(0.035)	(0.001)	[0.000]	[0.697]	[0.000]
10%	0.018	0.012	-0.514	0.155	0.075	0.051	0.869	0.260	14.893	2.940	56.938
		(0.000)	(0.063)		(0.025)	(0.274)	(0.048)	(0.017)	[0.000]	[0.003]	[0.000]
5%	0.025	0.011	-0.665	0.202	0.139	-0.470	0.901	0.200	5.321	0.731	26.387
		(0.000)	(0.074)		(0.081)	(0.530)	(0.060)	(0.038)	[0.000]	[0.465]	[0.000]
6.00%	0.024	0.008	-0.356	0.182	0.125	-0.331	0.916	0.167	6.139	0.074	36.674
6.10%		(0.000)	(0.127)		(0.057)	(0.394)	(0.052)	(0.027)	[0.000]	[0.941]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for the US equity markets represented by the S&P 500 index and bitcoin. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables: $p = p^{US} = p^{BTC}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0{:}\,\rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering two special cases, i.e., asymptotic independence $H_0{:}\,\rho = 0$ and total dependence $H_0{:}\,\rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 3A. Estimation of the bivariate distribution of return exceedances for the European equity market and gold

Panel A: Negative return exceedances

Parameters of the model									Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
5%	0.036	0.015	0.057	0.033	0.023	-0.260	0.965	0.060	62.609	61.615	977.029
		(0.005)	(0.280)		(0.007)	(0.244)	(0.033)	(0.001)	[0.000]	[0.000]	[0.000]
10%	0.029	0.010	0.267	0.025	0.015	-0.017	0.915	0.167	176.997	1.592	882.646
		(0.003)	(0.237)		(0.004)	(0.184)	(0.033)	(0.001)	[0.000]	[0.111]	[0.000]
20%	0.018	0.014	0.017	0.019	0.010	0.178	0.901	0.201	15.461	0.734	61.461
		(0.002)	(0.107)		(0.002)	(0.150)	(0.025)	(0.013)	[0.000]	[0.462]	[0.000]
30%	0.008	0.019	-0.099	0.011	0.016	-0.058	0.817	0.353	19.397	0.139	35.490
		(0.002)	(0.068)		(0.002)	(0.081)	(0.024)	(0.018)	[0.000]	[0.890]	[0.000]
40%	0.003	0.019	-0.087	0.004	0.019	-0.124	0.750	0.460	19.210	0.574	22.557
		(0.002)	(0.064)		(0.002)	(0.062)	(0.021)	(0.024)	[0.000]	[0.566]	[0.000]
50%	0.000	0.019	-0.075	0.000	0.021	-0.152	0.707	0.522	18.782	0.779	17.599
		(0.002)	(0.063)		(0.002)	(0.054)	(0.018)	(0.028)	[0.000]	[0.436]	[0.000]
3.01%	0.042	0.030	0.021	0.025	0.111	0.014	0.936	0.123	0.135	0.033	11.580
10.04%		(0.009)	(0.008)		(0.016)	(0.003)	(0.041)	(0.077)	[0.012]	[0.016]	[0.000]

Panel B: Positive return exceedances

Parameters of the model									Wald tests		
p	u^{EU}	σ^{EU}	ξ^{EU}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
50%	0.000	0.017	-0.151	0.000	0.021	-0.285	0.638	0.606	19.628	1.461	31.804
		(0.001)	(0.041)		(0.001)	(0.034)	(0.017)	(0.031)	(0.000)	(0.144)	(0.000)
40%	0.006	0.014	-0.094	0.006	0.016	-0.173	0.726	0.473	20.201	0.245	42.256
		(0.001)	(0.058)		(0.002)	(0.062)	(0.022)	(0.023)	(0.000)	(0.807)	(0.000)
30%	0.010	0.012	-0.050	0.012	0.012	-0.055	0.773	0.411	23.845	1.541	57.633
		(0.001)	(0.079)		(0.002)	(0.094)	(0.026)	(0.017)	(0.000)	(0.123)	(0.000)
20%	0.015	0.012	-0.067	0.024	0.012	-0.064	0.817	0.341	42.200	4.619	123.536
		(0.002)	(0.086)		(0.003)	(0.161)	(0.031)	(0.008)	(0.000)	(0.000)	(0.000)
10%	0.024	0.009	0.021	0.025	0.013	-0.118	0.801	0.366	44.033	7.837	120.099
		(0.002)	(0.132)		(0.003)	(0.159)	(0.044)	(0.008)	(0.000)	(0.000)	(0.000)
5%	0.031	0.008	0.094	0.033	0.009	0.111	0.796	0.372	12.308	12.280	32.711
		(0.002)	(0.203)		(0.003)	(0.344)	(0.061)	(0.030)	(0.000)	(0.000)	(0.000)
2.02%	0.040	0.003	0.668	0.028	0.011	-0.023	0.855	0.270	0.286	0.001	6.507
8.08%		(0.002)	(0.538)		(0.003)	(0.204)	(0.068)	(0.118)	[0.044]	[0.000]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for the European equity market, represented by the STOXX Europe 600 index, and gold. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{Gold}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering two special cases, i.e., asymptotic independence $H_0: \rho = 0$ and total dependence $H_0: \rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 3B. Estimation of the bivariate distribution of the return exceedances for the US equity market and gold

Panel A: Negative return exceedances

				Parameters of the model						Wald tests	
p	u^{US}	σ^{US}	ξ^{US}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
5%	0.031	0.026	-0.505	0.033	0.023	-0.260	0.966	0.089	9.888	2.842	101.222
		(0.008)	(0.278)		(0.007)	(0.244)	(0.037)	(0.009)	[0.000]	[0.004]	[0.000]
10%	0.023	0.013	0.058	0.025	0.015	-0.017	0.901	0.193	193.000	16.696	806.998
		(0.004)	(0.249)		(0.004)	(0.184)	(0.035)	(0.001)	[0.000]	[0.000]	[0.000]
20%	0.013	0.014	0.013	0.019	0.010	0.178	0.877	0.237	19.890	0.000	64.008
		(0.002)	(0.136)		(0.002)	(0.150)	(0.027)	(0.012)	[0.000]	[1.000]	[0.000]
30%	0.007	0.015	-0.013	0.011	0.016	-0.058	0.779	0.415	22.105	2.085	31.210
		(0.002)	(0.100)		(0.002)	(0.081)	(0.024)	(0.019)	[0.000]	[0.037]	[0.000]
40%	0.002	0.013	0.045	0.000	0.021	-0.162	0.708	0.515	19.214	0.214	18.119
		(0.002)	(0.087)		(0.002)	(0.053)	(0.019)	(0.027)	[0.000]	[0.830]	[0.000]
50%	0.000	0.013	0.035	0.000	0.021	-0.152	0.671	0.558	19.045	0.803	15.056
		(0.001)	(0.079)		(0.002)	(0.054)	(0.018)	(0.029)	[0.000]	[0.422]	[0.000]
6.06%	0.028	0.024	-0.394	0.025	0.015	-0.017	0.934	0.189	188.750	16.401	810.912
10.01%		(0.008)	(0.261)		(0.004)	(0.184)	(0.035)	(0.001)	[0.000]	[0.000]	[0.000]

Panel B: Positive return exceedances

				Parameters of the model						Wald tests	
p	u^{US}	σ^{US}	ξ^{US}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
50%	0.000	0.014	-0.122	0.000	0.021	-0.285	0.631	0.614	20.258	2.070	32.393
		(0.001)	(0.056)		(0.001)	(0.034)	(0.018)	(0.030)	[0.000]	[0.038]	[0.000]
40%	0.004	0.012	-0.055	0.006	0.016	-0.173	0.695	0.516	21.108	1.730	40.370
		(0.001)	(0.074)		(0.002)	(0.062)	(0.022)	(0.024)	[0.000]	[0.084]	[0.000]
30%	0.008	0.011	-0.012	0.012	0.012	-0.055	0.745	0.453	24.473	3.195	53.530
		(0.001)	(0.094)		(0.002)	(0.094)	(0.025)	(0.019)	[0.000]	[0.001]	[0.000]
20%	0.012	0.011	-0.008	0.016	0.014	-0.144	0.819	0.338	36.281	3.579	107.008
		(0.002)	(0.121)		(0.002)	(0.099)	(0.030)	(0.009)	[0.000]	[0.000]	[0.000]
10%	0.019	0.012	-0.072	0.025	0.013	-0.118	0.856	0.274	42.145	8.827	153.822
		(0.003)	(0.165)		(0.003)	(0.159)	(0.041)	(0.006)	[0.000]	[0.000]	[0.000
5%	0.027	0.012	-0.105	0.033	0.009	0.111	0.864	0.259	9.477	5.027	36.304
		(0.004)	(0.250)		(0.003)	(0.344)	(0.055)	(0.027)	[0.000]	[0.000]	[0.000]
2.70%	0.027	0.012	-0.105	0.029	0.012	-0.083	0.855	0.269	0.286	0.030	12.789
7.07%		(0.004)	(0.250)		(0.003)	(0.213)	(0.052)	(0.090)	[0.022]	[0.015]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for the US equity markets represented by the S&P 500 index and gold. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables, i.e., $p = p^{US} = p^{Gold}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering two special cases, i.e., asymptotic independence $H_0: \rho = 0$ and total dependence $H_0: \rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 4. Estimation of the bivariate distribution of return exceedances for bitcoin and gold

Panel A: Negative return exceedances

		Parameters of the model							Wald tests		
p	u^{BTC}	σ^{BTC}	ξ^{BTC}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
5%	0.193	0.073	-0.182	0.032	0.024	-0.563	0.949	0.083	3.934	0.698	43.317
		(0.045)	(0.555)		(0.013)	(0.561)	(0.047)	(0.021)	[0.000]	[0.485]	[0.000]
10%	0.149	0.077	-0.176	0.023	0.016	-0.155	0.973	0.049	5.061	2.663	99.065
		(0.022)	(0.219)		(0.005)	(0.250)	(0.026)	(0.010)	[0.000]	[0.008]	[0.000]
20%	0.075	0.112	-0.288	0.017	0.011	0.076	0.860	0.254	82.112	0.492	241.551
		(0.020)	(0.124)		(0.002)	(0.168)	(0.034)	(0.003)	[0.000]	[0.623]	[0.000]
30%	0.027	0.131	-0.312	0.010	0.015	-0.094	0.790	0.394	34.593	1.075	53.109
		(0.018)	(0.092)		(0.002)	(0.105)	(0.030)	(0.011)	[0.000]	[0.282]	[0.000]
40%	0.011	0.104	-0.166	0.004	0.018	-0.170	0.740	0.462	25.310	0.372	29.532
		(0.015)	(0.100)		(0.002)	(0.080)	(0.026)	(0.018)	[0.000]	[0.710]	[0.000]
50%	0.000	0.092	-0.086	0.000	0.020	-0.208	0.698	0.520	22.429	1.240	20.688
		(0.013)	(0.106)		(0.002)	(0.069)	(0.023)	(0.023)	[0.000]	[0.215]	[0.000]
11.03%	0.137	0.087	-0.231	0.022	0.014	-0.065	0.952	0.054	22.662	2.962	399.423
11.00%		(0.023)	(0.195)		(0.004)	(0.246)	(0.031)	(0.002)	[0.000]	[0.003]	[0.000]

Panel B: Positive return exceedances

		Parameters of the model							Wald tests		
p	u^{BTC}	σ^{BTC}	ξ^{BTC}	u^{Gold}	σ^{Gold}	ξ^{Gold}	α	ρ	$H_0: \rho = 0$	$H_0: \rho = \rho_{nor}^{f.s}(u)$	$H_0: \rho = 1$
50%	0.000	0.094	-0.042	0.000	0.018	-0.215	0.648	0.590	21.983	0.553	36.643
		(0.012)	(0.088)		(0.002)	(0.051)	(0.020)	(0.027)	[0.000]	[0.580]	[0.000
40%	0.024	0.099	-0.078	0.006	0.016	-0.192	0.722	0.492	27.353	1.224	55.076
		(0.013)	(0.095)		(0.002)	(0.062)	(0.026)	(0.018)	[0.000]	[0.221]	[0.000]
30%	0.050	0.099	-0.088	0.012	0.012	-0.104	0.781	0.385	35.816	0.792	92.725
		(0.016)	(0.111)		(0.002)	(0.089)	(0.031)	(0.011)	[0.000]	[0.428]	[0.000]
20%	0.089	0.095	-0.084	0.016	0.011	-0.089	0.829	0.331	296.471	2.485	896.122
		(0.019)	(0.142)		(0.002)	(0.112)	(0.036)	(0.001)	[0.000]	[0.013]	[0.000]
10%	0.155	0.075	0.051	0.024	0.009	0.030	0.919	0.164	15.118	0.393	91.890
		(0.025)	(0.274)		(0.002)	(0.205)	(0.041)	(0.011)	[0.000]	[0.694]	[0.000]
5%	0.202	0.139	-0.470	0.032	0.006	0.249	0.948	0.106	4.740	1.167	44.464
		(0.081)	(0.530)		(0.002)	(0.363)	(0.048)	(0.022)	[0.000]	[0.243]	[0.000]
10.34%	0.155	0.067	0.125	0.038	0.004	0.586	0.999	0.024	0.478	1.093	19.936
2.29%		(0.023)	(0.285)		(0.002)	(0.649)	(0.000)	(0.050)	[0.633]	[0.274]	[0.000]

Note: This table gives the asymptotic maximum likelihood estimates of the parameters of the bivariate distribution of return exceedances for bitcoin and gold. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with a threshold u. Both fixed and optimal threshold levels are used for u. The fixed levels correspond to tail probability p: 5%, 10%, 20%, 30%, 40% and 50% (the same value of p is taken for both variables, i.e., $p = p^{BTC} = p^{Gold}$). The optimal levels are computed by the procedure described in Appendix 4. They are given on the last line of each panel. Eight parameters are estimated, as follows: the threshold u associated with the tail probability p, the dispersion parameter σ, the tail index ξ for each series, the dependence parameter α of the logistic function used to model the tail dependence and the correlation of return exceedances ρ (derived from the dependence parameter α). Standard errors are given below in parentheses. The null hypothesis of normality $H_0: \rho = \rho_{nor}$ is also tested by a Wald test. Two cases are considered, i.e., the asymptotic case and the finite-sample case. In the asymptotic case, the correlation of normal return exceedances over a threshold tending to infinity is theoretically equal to 0. In the finite-sample case, the correlation of return exceedances over a given finite threshold u, denoted by $\rho_{nor}^{f.s.}(u)$, is computed by simulation, assuming that the returns follow a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The issue of dependency is studied by considering two special cases, i.e., asymptotic independence $H_0: \rho = 0$ and total dependence $H_0: \rho = 1$. The p-values of the Wald tests are given below in brackets.

Table 5. Comparative results for equity markets, bitcoin and gold

Panel A: Correlation among the return exceedances for the European equity market, bitcoin and gold

Negative return exceedances				Positive return exceedances			
Parameters			Wald test	Parameters			Wald test
p	$\rho^{EU/BTC}$	$\rho^{EU/Gold}$	$H_0: \rho^{EU/BTC} = \rho^{EU/Gold}$	p	$\rho^{EU/BTC}$	$\rho^{EU/Gold}$	$H_0: \rho^{EU/BTC} = \rho^{EU/Gold}$
5%	0.019	0.060	0.804	5%	0.084	0.372	5.647
	(0.050)	(0.001)	[0.421]		(0.021)	(0.030)	[0.000]
10%	0.170	0.167	0.333	10%	0.209	0.366	7.476
	(0.008)	(0.001)	[0.739]		(0.013)	(0.008)	[0.000]
20%	0.234	0.201	1.434	20%	0.265	0.341	6.909
	(0.010)	(0.013)	[0.1513]		(0.003)	(0.008)	[0.000]
30%	0.364	0.353	0.379	30%	0.353	0.411	2.071
	(0.011)	(0.018)	[0.704]		(0.011)	(0.017)	[0.038]
40%	0.472	0.460	0.286	40%	0.456	0.473	0.415
	(0.018)	(0.024)	[0.775]		(0.018)	(0.023)	[0.678]
50%	0.477	0.522	0.900	50%	0.609	0.606	0.051
	(0.022)	(0.028)	[0.368]		(0.028)	(0.031)	[0.959]
Optimal thresholds	0.164	0.123	0.482	Optimal thresholds	0.175	0.270	0.785
	(0.008)	(0.077)	[0.630]		(0.003)	(0.118)	[0.432]

Panel B: Correlation among the return exceedances for the US equity market, bitcoin and gold

Negative return exceedances				Positive return exceedances			
Parameters			Wald test	Parameters			Wald test
p	$\rho^{US/BTC}$	$\rho^{US/Gold}$	$H_0: \rho^{US/BTC} = \rho^{US/Gold}$	p	$\rho^{US/BTC}$	$\rho^{US/Gold}$	$H_0: \rho^{US/BTC} = \rho^{US/Gold}$
5%	0.123	0.089	1.619	5%	0.200	0.259	0.908
	(0.012)	(0.009)	[0.105]		(0.038)	(0.027)	[0.364]
10%	0.186	0.193	0.738	10%	0.260	0.274	0.609
	(0.011)	(0.001)	[0.333]		(0.017)	(0.006)	[0.543]
20%	0.333	0.237	7.385	20%	0.293	0.338	4.500
	(0.001)	(0.012)	[0.000]		(0.001)	(0.009)	[0.000]
30%	0.414	0.415	0.033	30%	0.312	0.453	4.862
	(0.011)	(0.019)	[0.973]		(0.010)	(0.019)	[0.000]
40%	0.499	0.469	0.714	40%	0.514	0.516	0.047
	(0.018)	(0.024)	[0.475]		(0.019)	(0.024)	[0.963]
50%	0.547	0.558	0.212	50%	0.599	0.614	0.268
	(0.023)	(0.029)	[0.832]		(0.026)	(0.030)	[0.789]
Optimal thresholds	0.192	0.189	0.886	Optimal thresholds	0.167	0.269	0.872
	(0.020)	(0.001)	[0.142]		(0.027)	(0.090)	[0.383]

Note: This table compares the results for equity markets, including bitcoin or gold. Panel A reports the correlation between return exceedances for the European equity market and bitcoin and the European equity market and gold. Panel B reports the correlation between the return exceedances for the US equity market and bitcoin and the US equity market and gold. For a given estimation, the same value of tail probability p is taken for the four variables, as follows: $p = p^{EU} = p^{US} = p^{BTC} = p^{Gold}$. Standard errors are given below in parentheses. The null hypotheses of equal correlation of return exceedances $H_0: \rho^{EU/BTC} = \rho^{EU/Gold}$ and $H_0: \rho^{US/BTC} = \rho^{US/Gold}$ are also tested by a Wald test. The p-values of the test are given below in brackets.

Table 6. Estimation of the bivariate distribution of return exceedances for the models of the logistic family

Panel A: Negative return exceedances

		Logistic		Asymmetric logistic		Negative logistic		Asymmetric negative logistic	
		$\chi(q)$	AIC	$\chi(q)$	AIC	$\chi(q)$	AIC	$\chi(q)$	AIC
Step 1: Equity markets	EU/US	0.764	**2.211**	0.533	17.982	0.769	2.748	0.476	12.579
Step 2: Equity markets and bitcoin	EU/BTC	0.090	60.071	0.030	64.184	0.002	**59.606**	0.023	64.190
	US/BTC	0.113	90.476	0.047	94.304	0.043	**90.335**	0.035	93.215
Step 3: Equity markets and gold	EU/Gold	0.079	39.864	0.000	42.284	0.000	**38.211**	0.000	42.210
	US/Gold	0.075	34.961	0.000	38.929	0.012	**34.842**	0.012	38.842
Step 4: Bitcoin and gold	BTC/Gold	0.050	**86.087**	0.000	89.081	0.001	86.100	0.000	89.042

Panel B: Positive return exceedances

		Logistic		Asymmetric logistic		Negative logistic		Asymmetric negative logistic	
		$\chi(q)$	AIC	$\chi(q)$	AIC	$\chi(q)$	AIC	$\chi(q)$	AIC
Step 1: Equity markets	EU/US	0.274	34.019	0.304	38.594	0.338	**33.849**	0.345	37.850
Step 2: Equity markets and bitcoin	EU/BTC	0.126	82.800	0.000	**81.829**	0.280	91.451	0.212	94.714
	US/BTC	0.109	**66.039**	0.132	75.525	0.111	75.620	0.021	69.839
Step 3: Equity markets and gold	EU/Gold	0.193	-1.973	0.221	3.170	0.287	**-2.795**	0.335	4.940
	US/Gold	0.194	26.776	0.149	30.848	0.146	**26.025**	0.167	30.168
Step 4: Bitcoin and gold	BTC/Gold	0.000	**108.213**	0.000	112.615	0.004	108.726	0.023	113.129

Note: This table gives the asymptotic maximum likelihood estimate of the quantile dependence parameter of return exceedances $\chi(q)$ across the four extreme value models of the logistic family. These models are as follows: the logistic, the asymmetric logistic, the negative logistic and the asymmetric negative logistic models. Panel A reports the estimates for the negative return exceedances. Panel B reports the estimates for the positive return exceedances. The return exceedances are defined with optimal threshold levels computed by the procedure described in Appendix 4. The corresponding Akaike information criterion (AIC) for each model is also computed. The parameter $\chi(q)$ measures the strength of quantile dependence across all the models of the logistic family. The special cases where $\chi(q)$ is equal to 1 and $\chi(q)$ is equal to 0 correspond to asymptotic independence and total dependence, respectively. The quantile q at optimal thresholds is defined as the corresponding tail probability p for negative return exceedances and $(1-p)$ for positive return exceedances. The minimum value of the AIC across the four models is highlighted in bold.

Table 7. Optimal asset allocation based on tail risk measures (two-asset portfolios)

Panel A: Tail risk defined by the Value at Risk (VaR)

		$p = 95\%$					$p = 99\%$					$p = 99.9\%$				
		Weights	VaR	$E(r)$	TP ratio	DP ratio	Weights	VaR	$E(r)$	TP ratio	DP ratio	Weights	VaR	$E(r)$	TP ratio	DP ratio
Step 1	EU/US	(30, 70)	3.25%	0.17%	0.012	0.031	(58, 42)	5.74%	0.14%	0.003	0.028	(28, 72)	9.20%	0.17%	0.051	0.053
Step 2	EU/BTC	(92, 8)	3.52%	0.21%	0.117	0.389	(89, 11)	5.95%	0.26%	0.219	0.445	(85, 15)	8.80%	0.32%	0.420	0.541
	US/BTC	(93, 7)	3.25%	0.30%	0.183	0.378	(88, 12)	5.75%	0.37%	0.637	0.510	(84, 16)	8.49%	0.42%	0.589	0.547
Step 3	EU/Gold	(93, 7)	3.47%	0.10%	0.051	0.060	(97, 3)	5.81%	0.10%	0.012	0.036	(95, 5)	9.32%	0.09%	0.022	0.069
	US/Gold	(91, 9)	3.27%	0.18%	0.074	0.081	(97, 3)	5.76%	0.19%	0.024	0.029	(98, 2)	8.95%	0.20%	0.015	0.009
Step 4	BTC/Gold	(4, 96)	3.20%	0.03%	0.203	0.219	(10, 90)	5.08%	0.13%	0.206	0.502	(16, 84)	7.80%	0.22%	0.211	0.619

Panel B: Tail risk defined by the Expected Shortfall (ES)

		$p = 95\%$					$p = 99\%$					$p = 99.9\%$				
		Weights	ES	$E(r)$	TP ratio	DP ratio	Weights	ES	$E(r)$	TP ratio	DP ratio	Weights	ES	$E(r)$	TP ratio	DP ratio
Step 1	EU/US	(42, 58)	4.73%	0.16%	0.024	0.038	(48, 52)	7.13%	0.15%	0.034	0.056	(28, 72)	10.68%	0.17%	0.201	0.054
Step 2	EU/BTC	(91, 9)	4.77%	0.23%	0.113	0.407	(87, 13)	7.27%	0.29%	0.169	0.480	(81, 19)	10.31%	0.38%	0.760	0.597
	US/BTC	(92, 8)	4.52%	0.31%	0.260	0.246	(85, 15)	7.18%	0.41%	0.980	0.572	(81, 19)	9.84%	0.46%	0.606	0.572
Step 3	EU/Gold	(95, 5)	4.96%	0.09%	0.005	0.056	(95, 5)	7.22%	0.10%	0.038	0.090	(94, 6)	10.70%	0.09%	0.020	0.131
	US/Gold	(96, 4)	4.79%	0.19%	0.032	0.036	(96, 4)	7.08%	0.19%	0.031	0.038	(98, 2)	10.30%	0.20%	0.015	0.013
Step 4	BTC/Gold	(8, 92)	4.33%	0.09%	0.206	0.409	(13, 87)	6.23%	0.17%	0.187	0.584	(19, 81)	9.17%	0.27%	0.262	0.614

Note: This table gives the optimal asset allocation for international equity markets, including either bitcoin or gold for Equal-VaR or Equal-ES portfolios. Step 1 refers to European and US equity markets. Step 2 refers to equity markets and bitcoin, i.e., the European equity market and bitcoin and the US equity market and bitcoin. Step 3 refers to equity markets and gold, i.e., the European equity market and gold and the US equity market and gold. Step 4 refers to bitcoin and gold. Panel A reports the optimal allocation, the Value at Risk (VaR) as a tail risk measure and the portfolio performance measures. Panel B reports the optimal allocation, the Expected Shortfall (ES) as a tail risk measure and the performance measures. The VaR and ES are estimated using the GPD, as defined in subsection 3.1 for different probability levels p: 95%, 99% and 99.9%. In addition to the expected return $E(r)$, we consider the tail performance (TP) and diversification performance (DP) ratios. For the same level of risk defined with either the VaR or ES (reflecting negative return exceedances in the left tail), the TP ratio is computed by the ratio between the upside quantile of the distribution of the new position (reflecting positive return exceedances in the right tail) and the upside quantile of the initial position. For the same level of risk defined with either the VaR or ES (reflecting negative return exceedances in the left tail), the DP ratio is computed by the ratio between the tail risk value (VaR or ES) of the new position with the empirical extreme correlation and the theoretical tail risk value estimated by simulation from a bivariate logistic distribution with parameters equal to the estimated parameters of returns, assuming total dependence of the negative extremes. The procedure for the computation of the optimal weights is described in Appendix 5.

Table 8. Optimal asset allocation based on tail risk measures (three-asset portfolios)

Panel A: Tail risk defined by the Value at Risk (VaR)

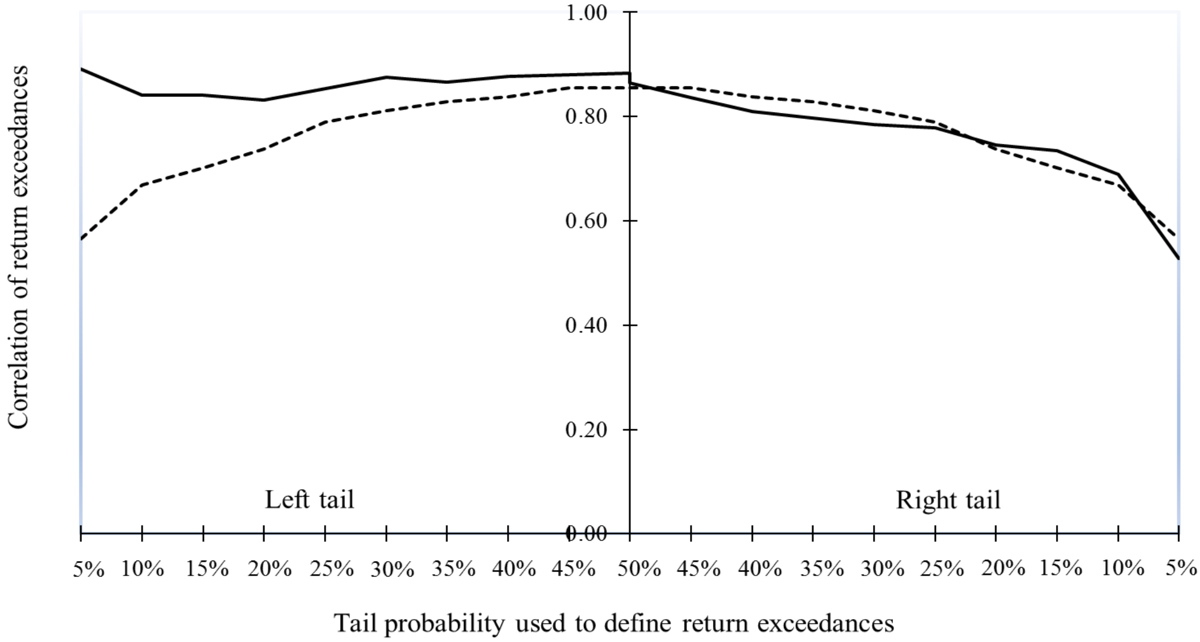
	$p = 95\%$					$p = 99\%$					$p = 99.9\%$				
	Weights	VaR	$E(r)$	TP ratio	DP ratio	Weights	VaR	$E(r)$	TP ratio	DP ratio	Weights	VaR	$E(r)$	TP ratio	DP ratio
EU/BTC/Gold	(64,15,21)	3.52%	0.29%	0.247	0.384	(55,17,28)	5.92%	0.31%	0.304	0.397	(59,19,22)	8.86%	0.35%	0.356	0.407
US/BTC/Gold	(75,11,14)	3.24%	0.32%	0.250	0.366	(62,16,22)	5.75%	0.37%	0.473	0.430	(53,19,28)	8.57%	0.44%	0.480	0.405

Panel B: Tail risk defined by the Expected Shortfall (ES)

	$p = 95\%$					$p = 99\%$					$p = 99.9\%$				
	Weights	ES	$E(r)$	TP ratio	DP ratio	Weights	ES	$E(r)$	TP ratio	DP ratio	Weights	ES	$E(r)$	TP ratio	DP ratio
EU/BTC/Gold	(62,15,23)	4.77%	0.29%	0.253	0.395	(61,20,19)	7.22%	0.37%	0.318	0.410	(53,25,22)	10.20%	0.39%	0.340	0.422
US/BTC/Gold	(65,13,22)	4.52%	0.33%	0.384	0.387	(70,19,11)	7.17%	0.44%	0.519	0.410	(53,26,21)	9.85%	0.51%	0.582	0.445

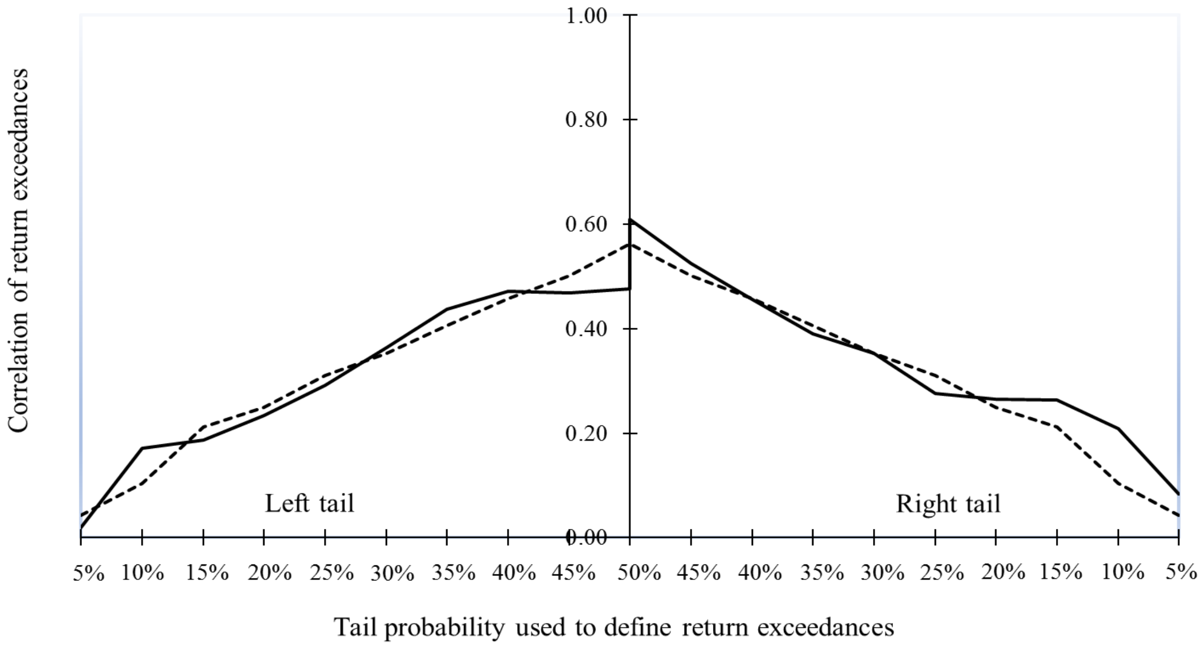
Note: This table gives the optimal asset allocation for international equity markets, including bitcoin and gold for Equal-VaR or Equal-ES portfolios, i.e., the European equity market, bitcoin and gold and the US equity market, bitcoin and gold. Panel A reports the optimal allocation, the Value at Risk (VaR) as a tail risk measure and the portfolio performance measures. Panel B reports the optimal allocation, the Expected Shortfall (ES) as a tail risk measure and the performance measures. The VaR and ES are estimated using the GPD, as defined in subsection 3.1 for different probability levels p: 95%, 99% and 99.9%. In addition to the expected return $E(r)$, we consider the tail performance (TP) and diversification performance (DP) ratios. For the same level of risk defined with either the VaR or ES (reflecting negative return exceedances in the left tail), the TP ratio is computed by the ratio between the upside quantile of the distribution of the new position (reflecting positive return exceedances in the right tail) and the upside quantile of the initial position. For the same level of risk defined with either the VaR or ES (reflecting negative return exceedances in the left tail), the DP ratio is computed by the ratio between the tail risk value (VaR or ES) of the new position with the empirical extreme correlation and the theoretical tail risk value estimated by simulation from a trivariate logistic distribution with parameters equal to the estimated parameters of returns, assuming total dependence of the negative extremes. The procedure for the computation of the optimal weights is described in Appendix 5.

Figure 1. Correlation between the return exceedances for the European and US equity markets



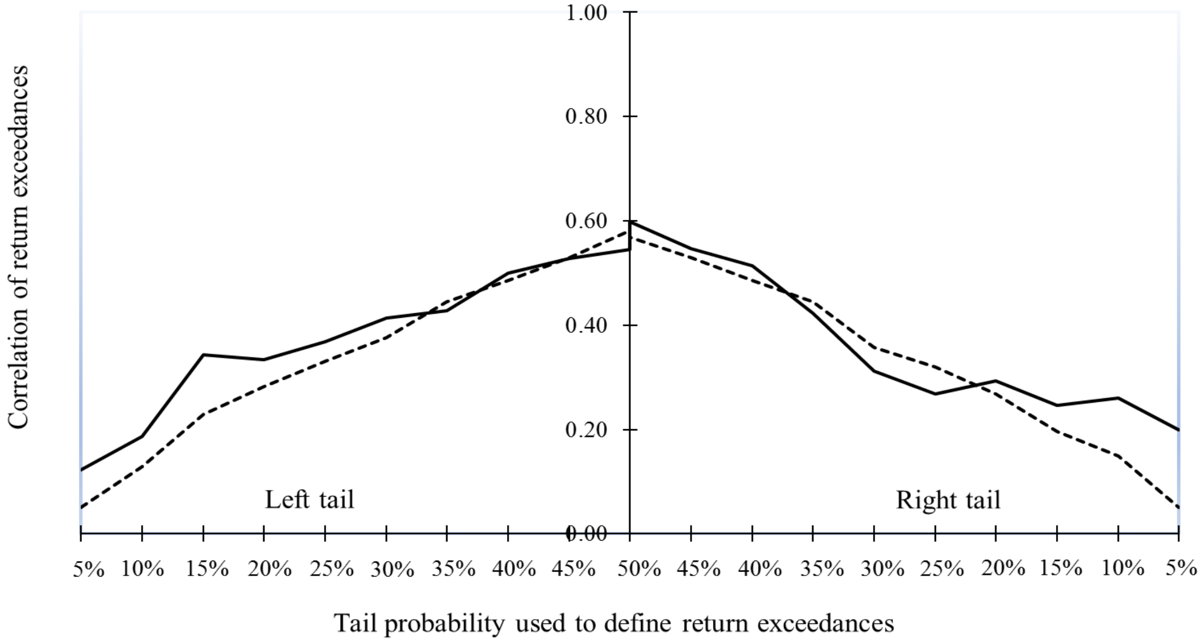
Note: This figure represents the correlation of return exceedances between the European and US equity markets represented by the STOXX Europe 600 index and the S&P 500 index. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 1). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances assuming a bivariate normal distribution with the parameters equal to the empirically observed means and the covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedance ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{US}$.

Figure 2A. Correlation between the return exceedances for the European equity market and bitcoin



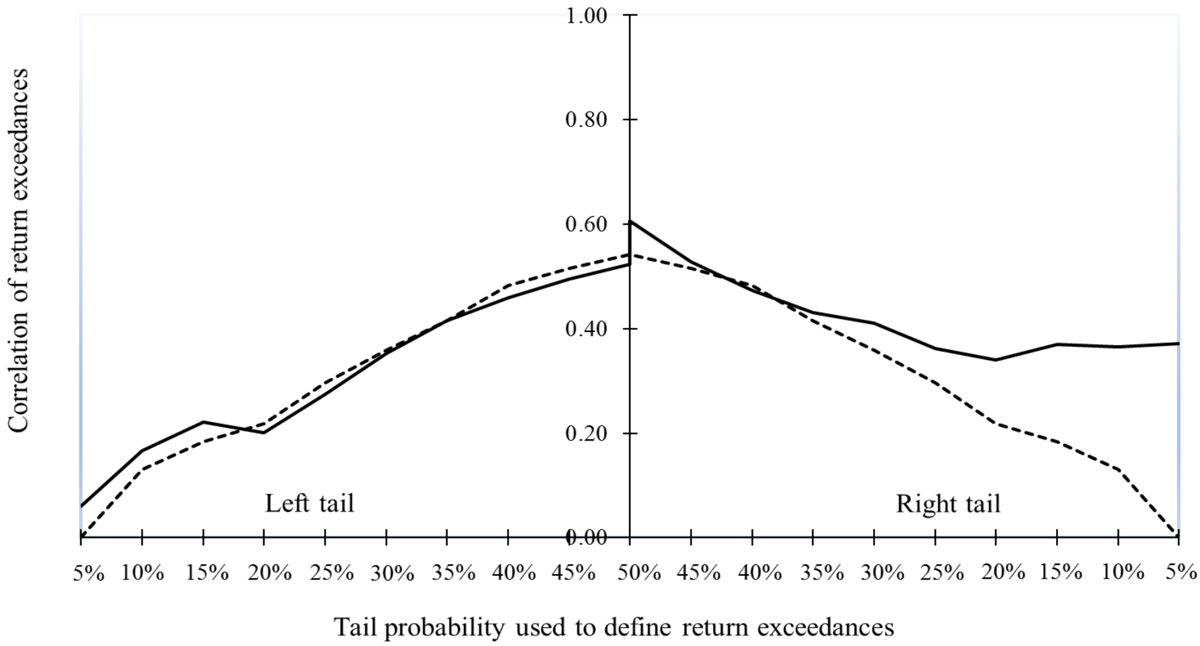
Note: This figure represents the correlation of the return exceedances between the European equity markets represented by the STOXX Europe 600 index and bitcoin. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 2A). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{BTC}$.

Figure 2B. Correlation between the return exceedances for the US equity market and bitcoin



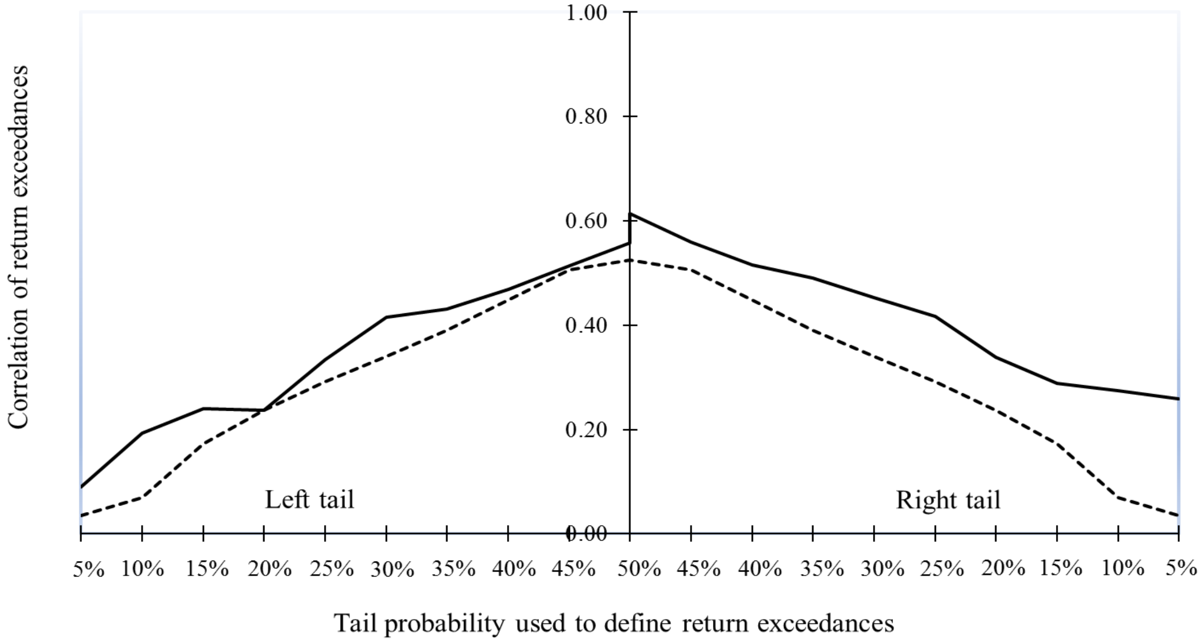
Note: This figure represents the correlation of the return exceedances between the US equity markets represented by the S&P 500 index and bitcoin. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 2B). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{US} = p^{BTC}$.

Figure 3A. Correlation between the return exceedances for the European equity market and gold



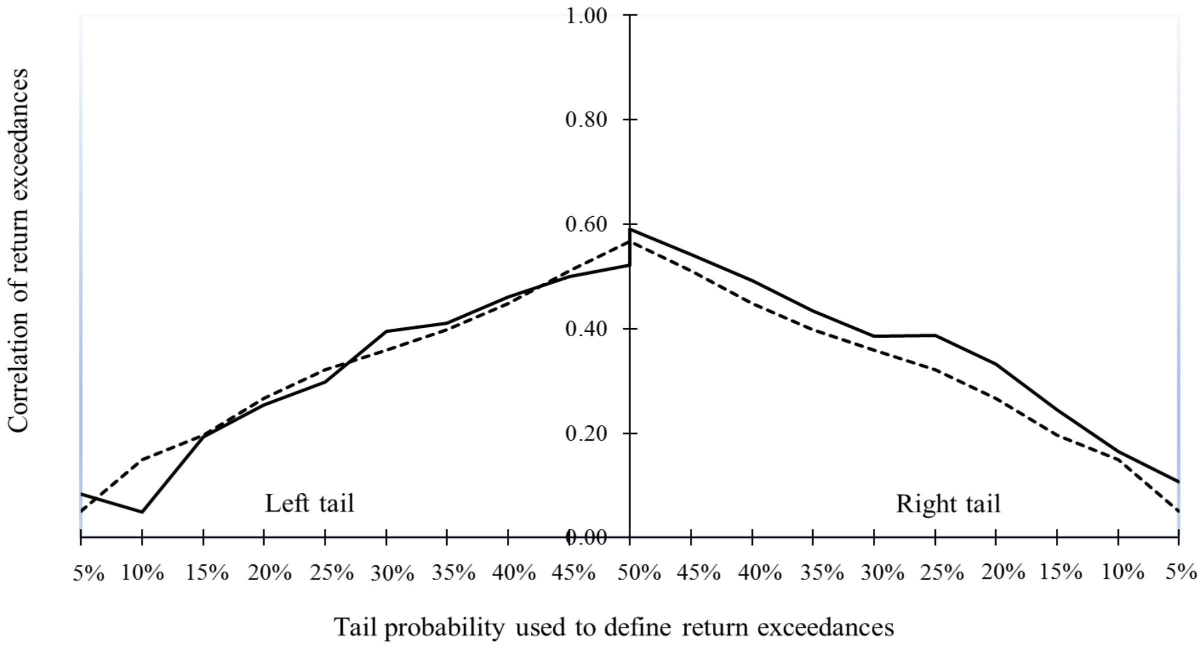
Note: This figure represents the correlation of return exceedances between the European equity markets represented by the STOXX Europe 600 index and gold. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 3A). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{EU} = p^{Gold}$.

Figure 3B. Correlation between the return exceedances for the US equity market and gold



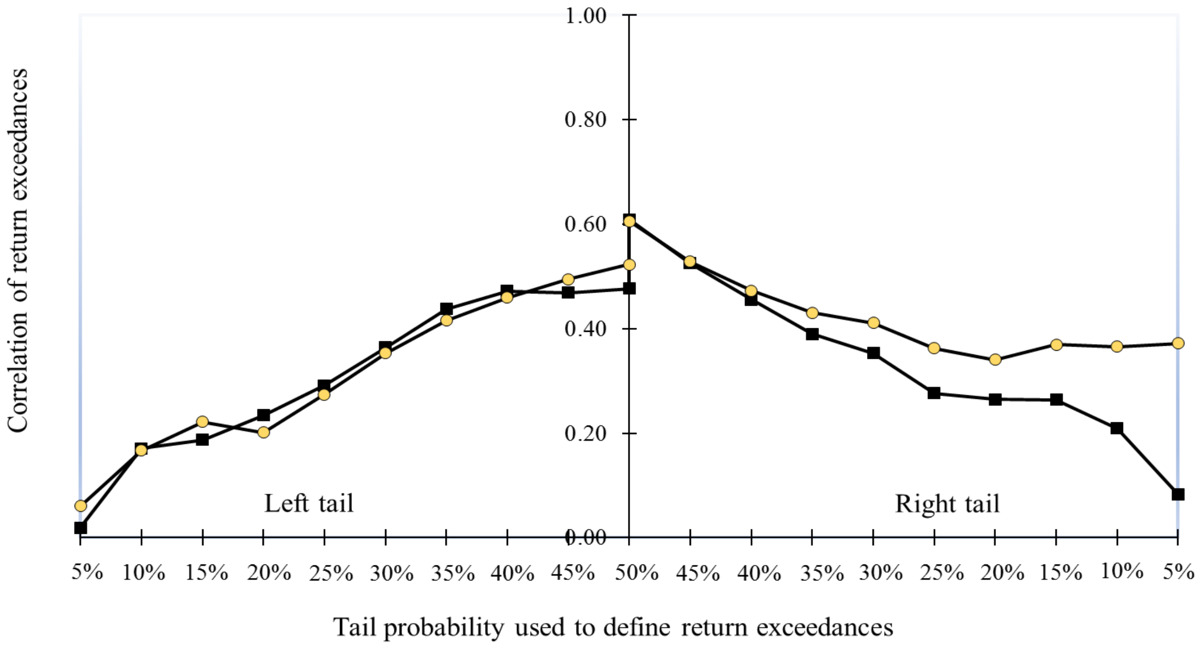
Note: This figure represents the correlation of the return exceedances between the US equity markets represented by the S&P 500 index and gold. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 3B). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{US} = p^{Gold}$.

Figure 4. Correlation between the return exceedances for bitcoin and gold



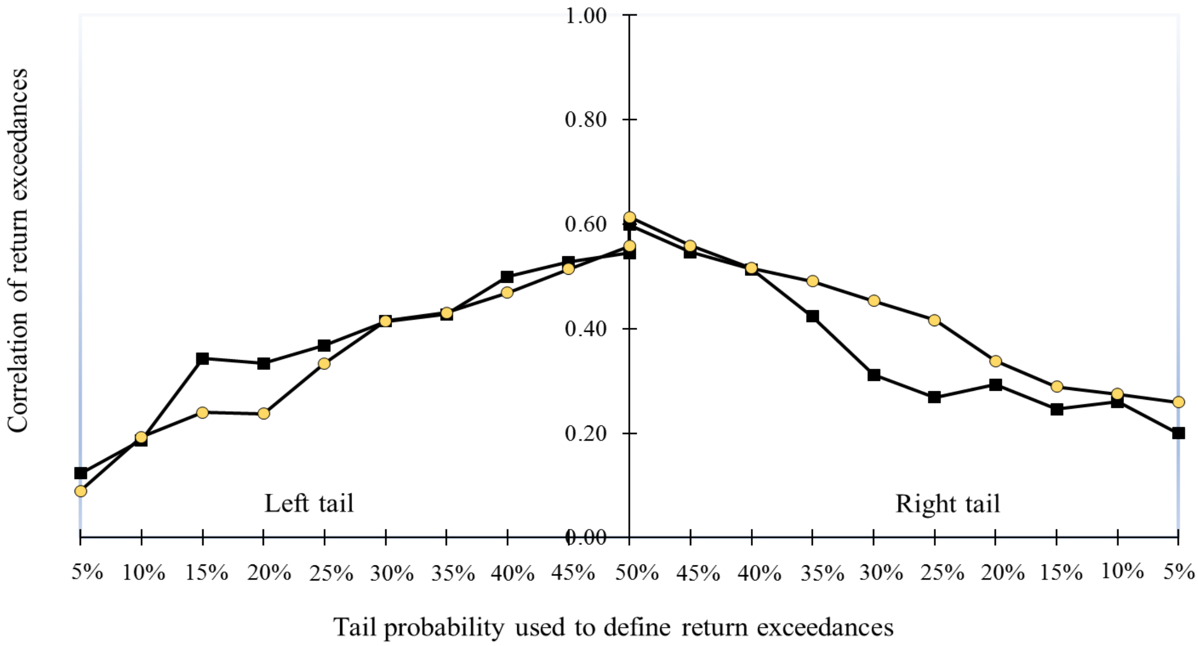
Note: This figure represents the correlation of the return exceedances between bitcoin and gold. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function (see the estimation results in Table 4). The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the threshold for the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for both variables, i.e., $p = p^{BTC} = p^{Gold}$.

Figure 5A. Correlation between the return exceedances for the European equity market, bitcoin or gold



Note: This figure represents the correlation of the return exceedances for the European equity market, with either bitcoin or gold (see the estimation results in Table 5 - Panel A). The *squared points line* represents the correlation between the return exceedances for the European equity market and bitcoin. The *circle points line* represents the correlation between the return exceedances for the European equity market and gold. The value of the tail probability p used to define the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for three variables, i.e., $p = p^{EU} = p^{BTC} = p^{Gold}$.

Figure 5B. Correlation between the return exceedances for the US equity market, bitcoin or gold



Note: This figure represents the correlation of the return exceedances for the US equity market, with either bitcoin or gold (see the estimation results in Table 5 - Panel B). The *squared points line* represents the correlation between the return exceedances for the US equity market and bitcoin. The *circle points line* represents the correlation between the return exceedances for the US equity market and gold. The value of the tail probability p used to define the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for three variables, i.e., $p = p^{US} = p^{BTC} = p^{Gold}$.

Figure 6. Correlation between the return exceedances for other international equity markets, bitcoin or gold

Figure 6A. Chinese equity market, bitcoin or gold

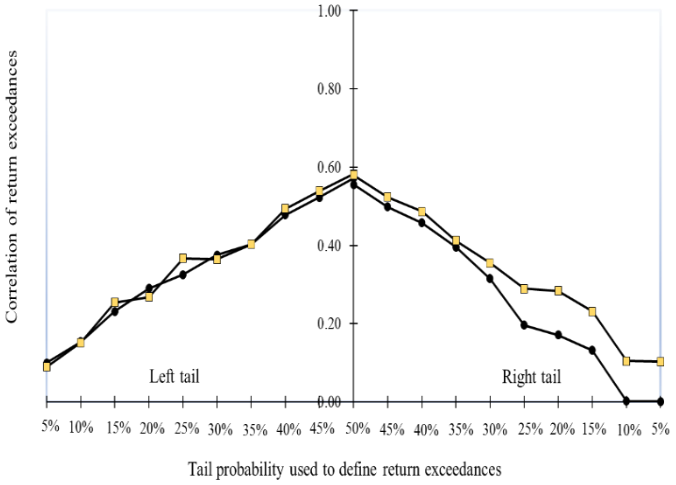


Figure 6B. Japanese equity market, bitcoin or gold

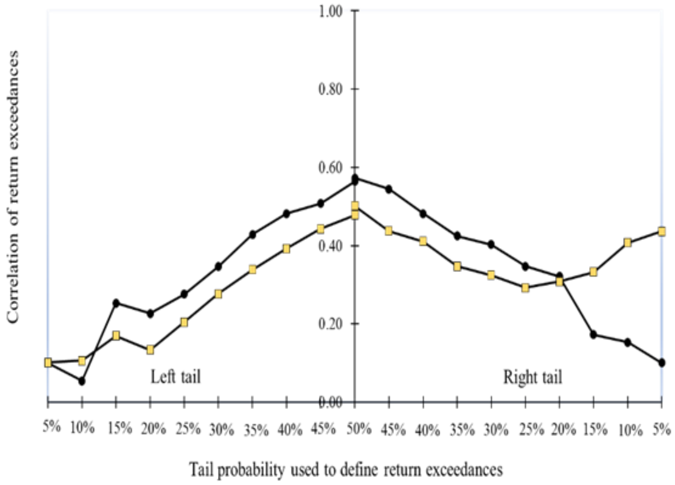


Figure 6C. French equity market, bitcoin or gold

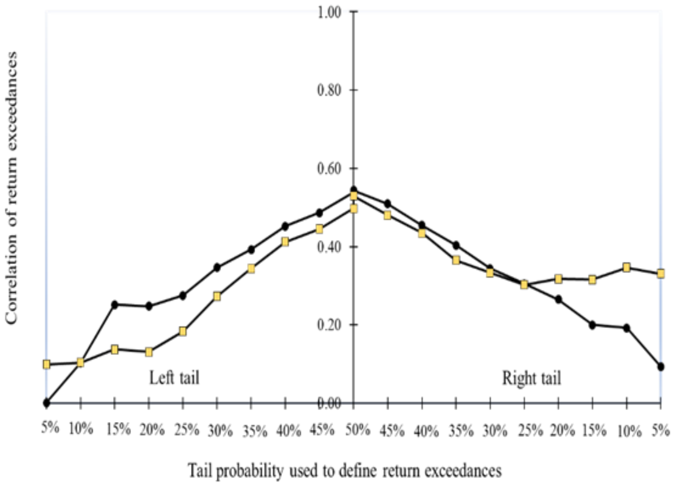


Figure 6D. German equity market, bitcoin or gold

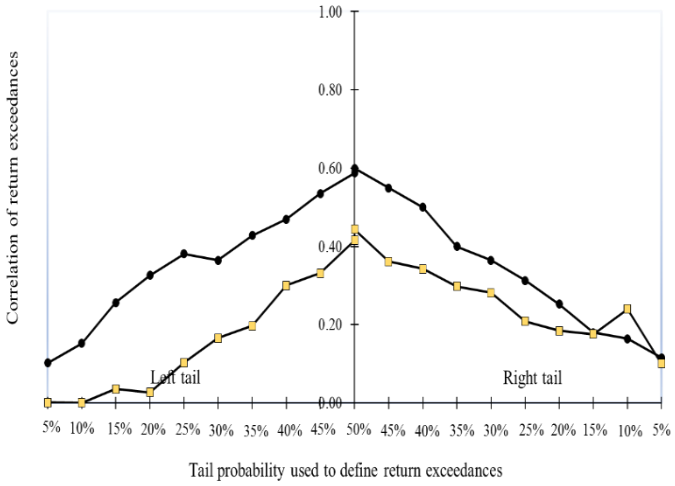
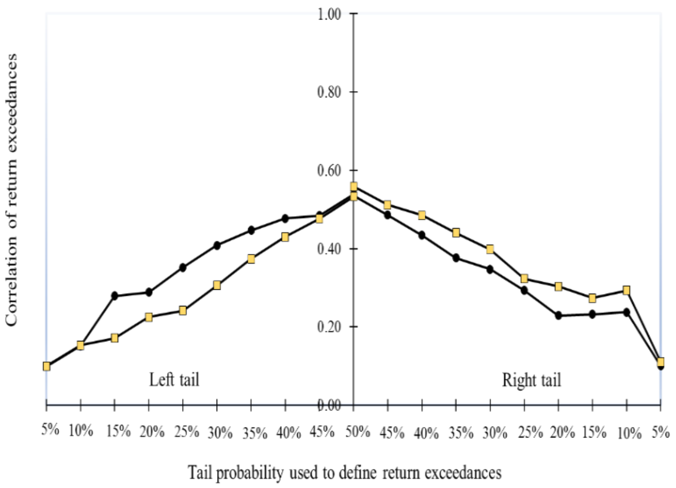


Figure 6E. UK equity market, bitcoin or gold



Note: This figure represents the correlation of the return exceedances for other international equity markets, with either bitcoin or gold. Figure 6A refers to the Chinese equity market, Figure 6B to the Japanese equity market, Figure 6C to the French equity market, Figure 6D to the German equity market and Figure 6E to the UK equity market, which are represented by the SSE 180, Nikkei 225, CAC 40, DAX 30 and FTSE 100 equity indices, respectively. The *squared points line* represents the correlation between the return exceedances for each equity market and bitcoin. The *circle points line* represents the correlation between the return exceedances for each equity market and gold. The value of the tail probability p used to define the return exceedances ranges from 5% to 50% for both the negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for three variables, i.e., $p = p^{EM} = p^{BTC} = p^{Gold}$, where EM stands for the equity markets.

Figure 7. Correlation between the return exceedances for the European and US equity markets, bitcoin, m-CRIX or gold

Figure 7A. Bitcoin and m-CRIX

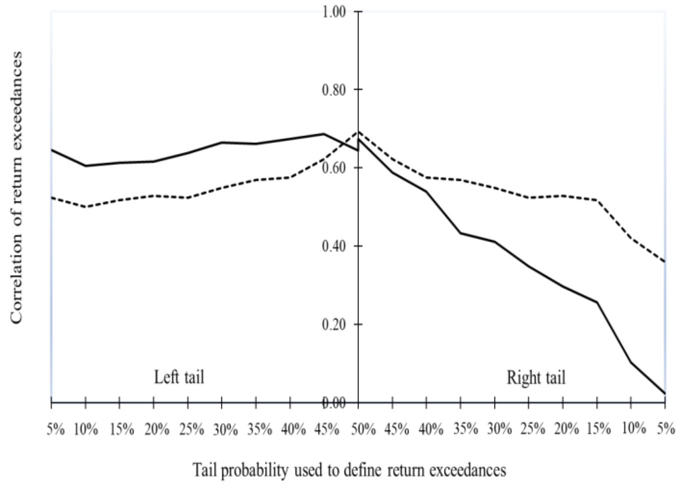


Figure 7B. European equity market and m-CRIX

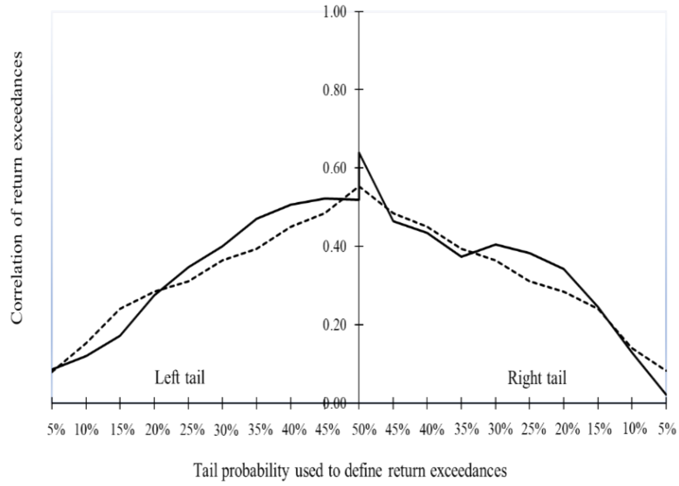


Figure 7C. US equity market and m-CRIX

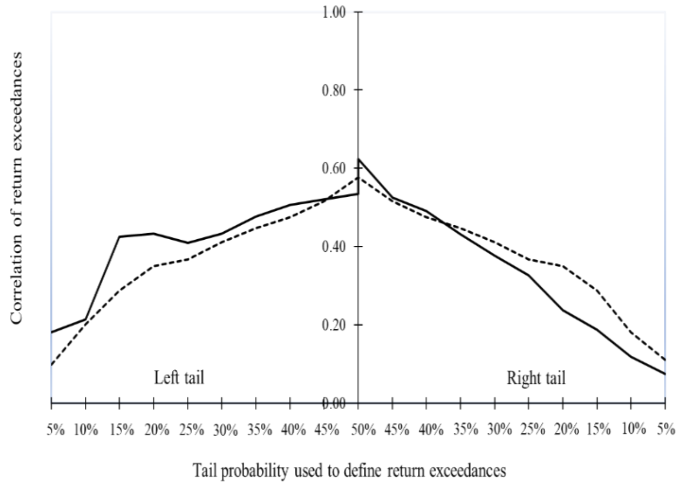
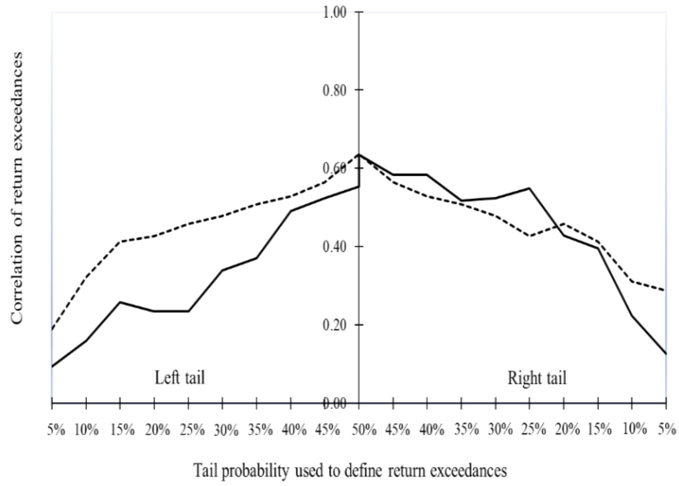


Figure 7D. m-CRIX and gold



Note: This figure represents the correlation of the return exceedances among the European and US equity markets (represented by the STOXX Europe 600 index and the S&P 500 index), with bitcoin, modified CRIX (m-CRIX) or gold. Figure 7A refers to the extreme correlation between bitcoin and the m-CRIX index; Figure 7B refers to the European equity market and m-CRIX index; Figure 7C refers to the US equity market and m-CRIX and Figure 7D refers to m-CRIX and gold. The *solid line* represents the correlation between the actual return exceedances obtained from the estimation of the bivariate distribution modeled with the logistic function. The *dotted line* represents the theoretical correlation between the simulated normal return exceedances, assuming a bivariate normal distribution, with parameters equal to the empirically observed means and covariance matrix of returns. The value of the tail probability p used to define the return exceedances ranges from 5% to 50% for both negative return exceedances (left tail) and positive return exceedances (right tail). For a given estimation, the same value of p is taken for all variables, i.e., $p = p^{EU} = p^{US} = p^{BTC} = p^{m-CRIX} = p^{Gold}$.

Figure 8. Behavior over time for the negative return exceedances

Figure 8A. Behavior over time for the European and US equity markets

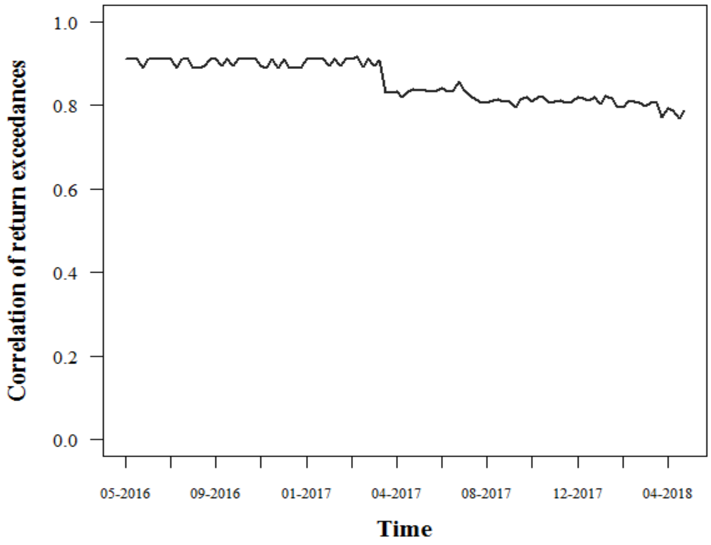


Figure 8B. Behavior over time for the European equity market and bitcoin

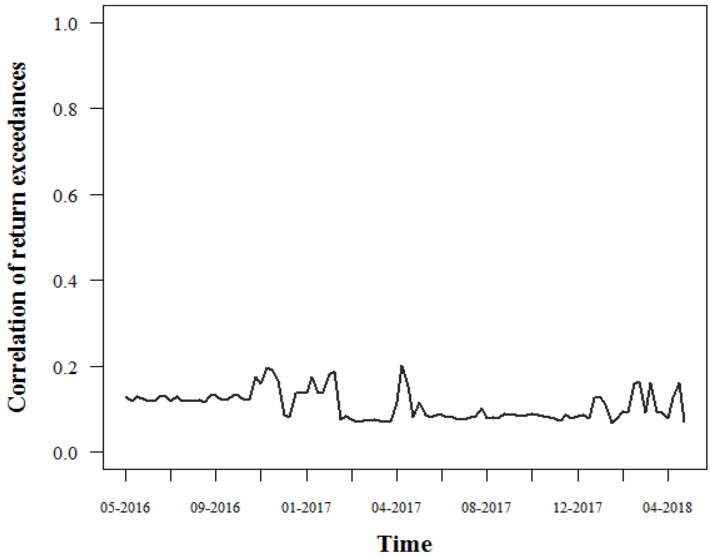


Figure 8C. Behavior over time for the US equity market and bitcoin

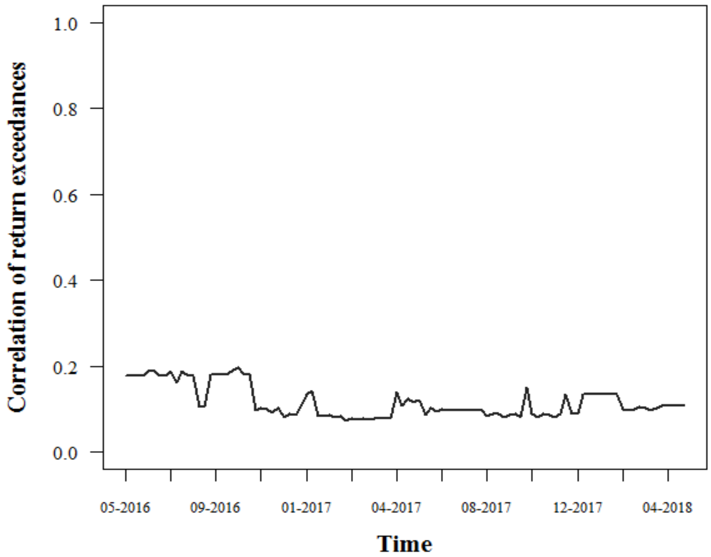


Figure 8D. Behavior over time for the European equity market and gold

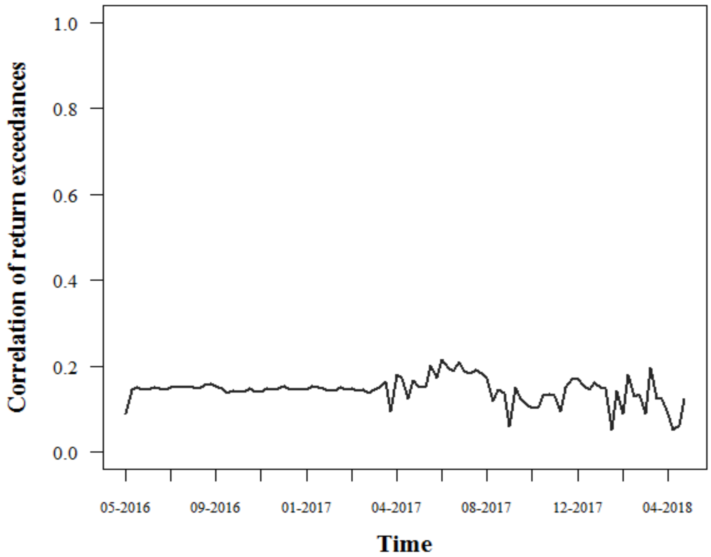


Figure 8E. Behavior over time for the US equity market and gold

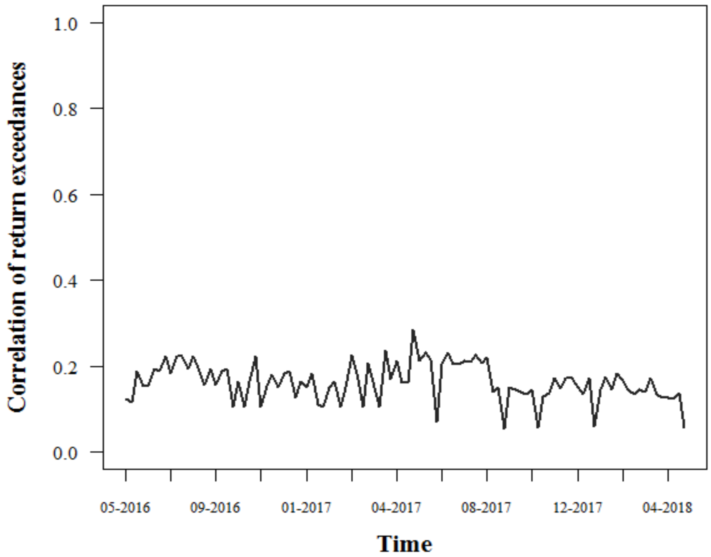
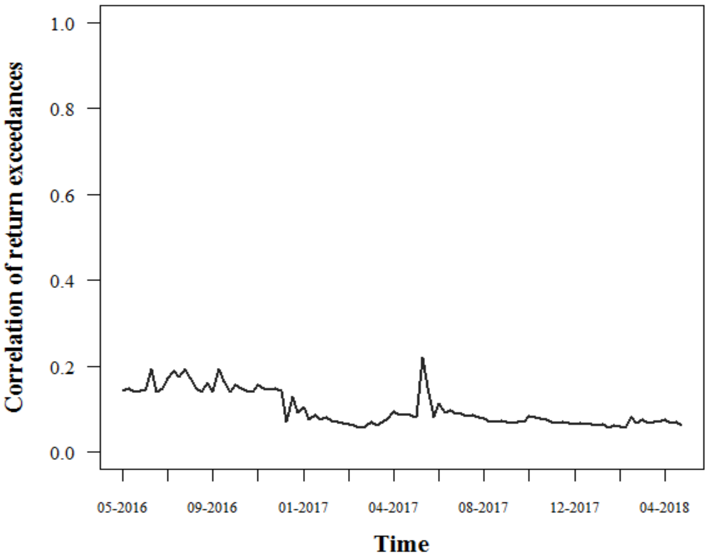


Figure 8F. Behavior over time for bitcoin and gold



Note: This figure represents the behavior over time of the extreme correlation for the negative return exceedances via a rolling estimation window for the European and US equity markets (represented by the STOXX Europe 600 index and the S&P 500 index) with either bitcoin or gold. Figure 8A refers to equity markets, i.e., European and US equity markets (Step 1). Figures 8B and 8C refer to equity markets and bitcoin (Step 2), i.e., the European equity market and bitcoin and the US equity market and bitcoin, respectively. Figures 8D and 8E refer to equity markets and gold (Step 3), i.e., the European equity market and gold and the US equity market and gold, respectively. Figure 8F refers to bitcoin and gold (Step 4).

Figure 9. Behavior over time for the positive return exceedances

Figure 9A. Behavior over time for the European and US equity markets

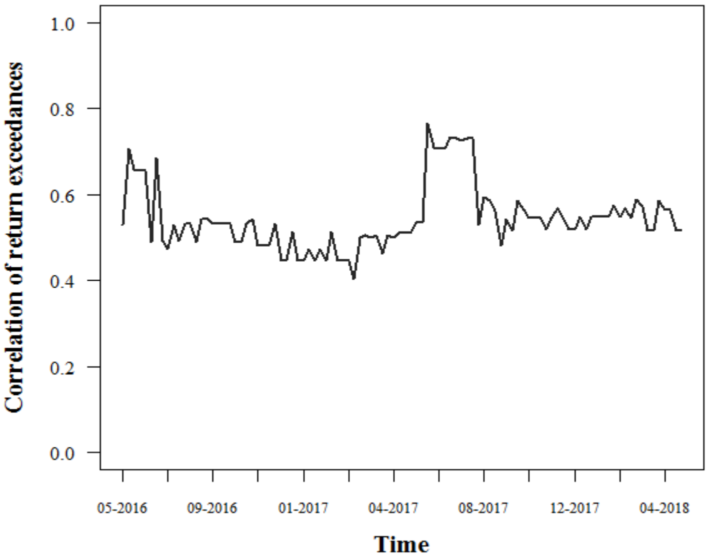


Figure 9B. Behavior over time for the European equity market and bitcoin

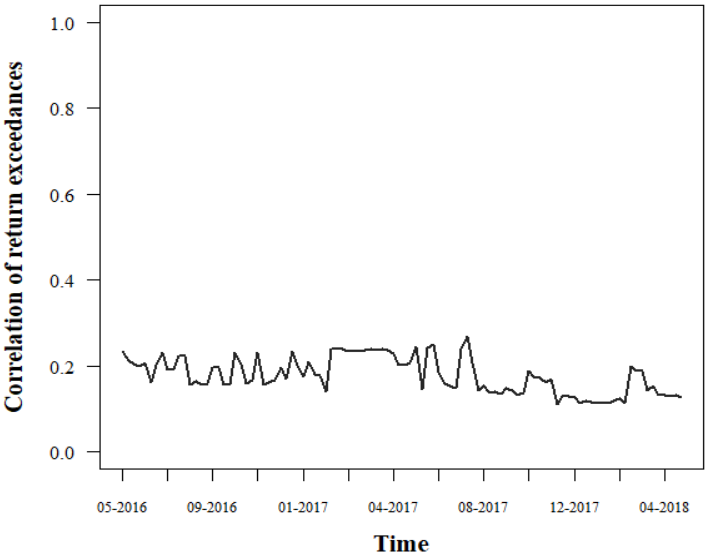


Figure 9C. Behavior over time for the US equity market and bitcoin

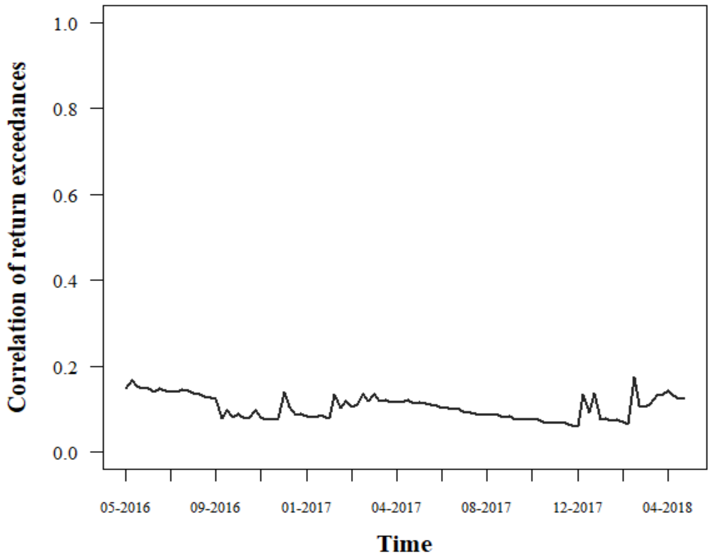


Figure 9D. Behavior over time for the European equity market and gold

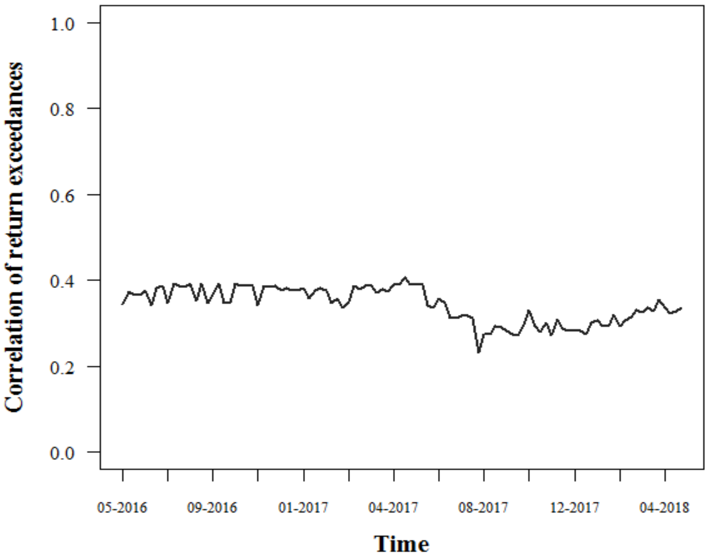


Figure 9E. Behavior over time for the US equity market and gold

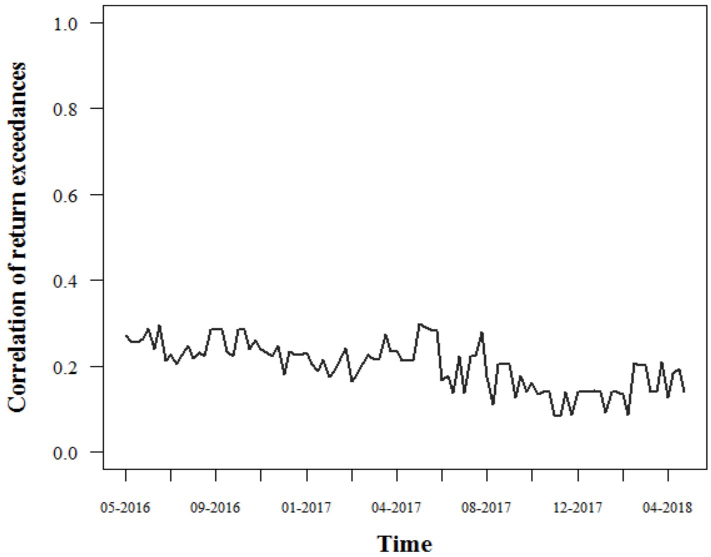
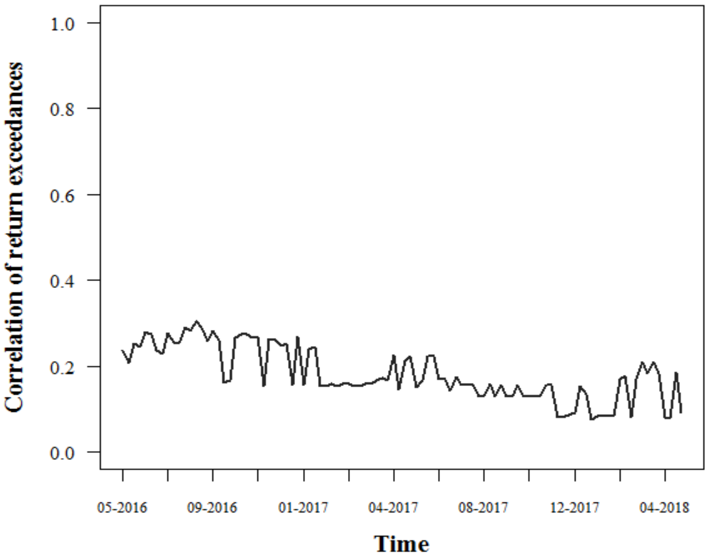


Figure 9F. Behavior over time for bitcoin and gold



Note: This figure represents the behavior over time of the extreme correlation for the positive return exceedances via a rolling estimation window for the European and US equity markets (represented by the STOXX Europe 600 index and the S&P 500 index) with either bitcoin or gold. Figure 9A refers to equity markets, i.e., European and US equity markets (Step 1). Figures 9B and 9C refer to equity markets and bitcoin (Step 2), i.e., the European equity market and bitcoin and the US equity market and bitcoin, respectively. Figures 9D and 9E refer to equity markets and gold (Step 3), i.e., the European equity market and gold and the US equity market and gold, respectively. Figure 9F refers to bitcoin and gold (Step 4).

Figure 10. Nonparametric copulas for the European and US equity markets with either bitcoin or gold

Figure 10A. Nonparametric copula for the European and US equity markets

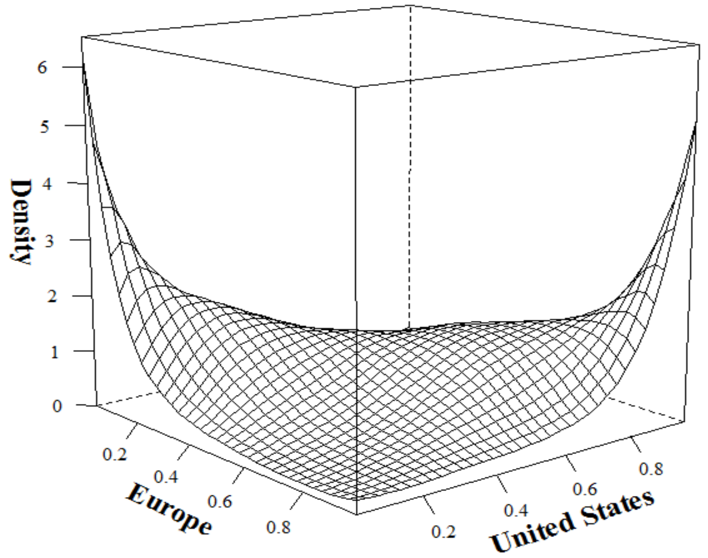


Figure 10B. Nonparametric copula for the European equity market and bitcoin

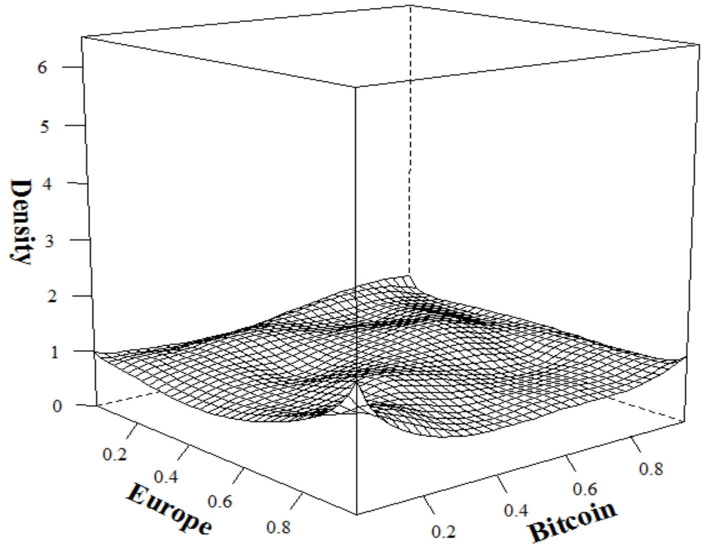


Figure 10C. Nonparametric copula for the US equity market and bitcoin

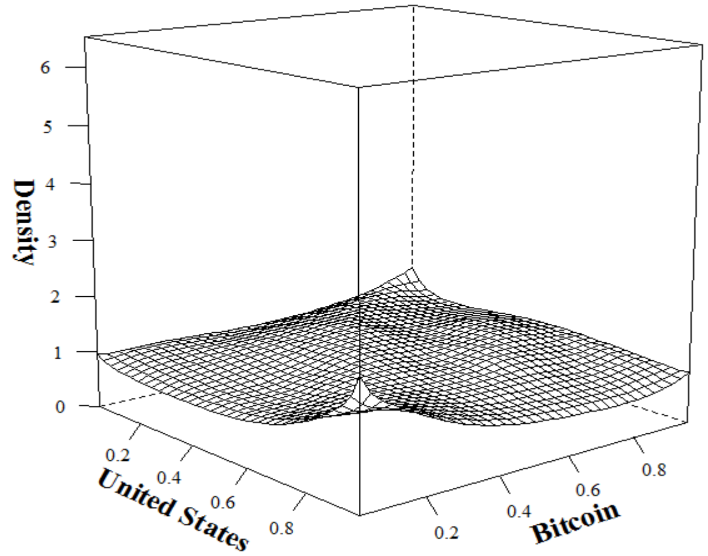


Figure 10D. Nonparametric copula for the European equity market and gold

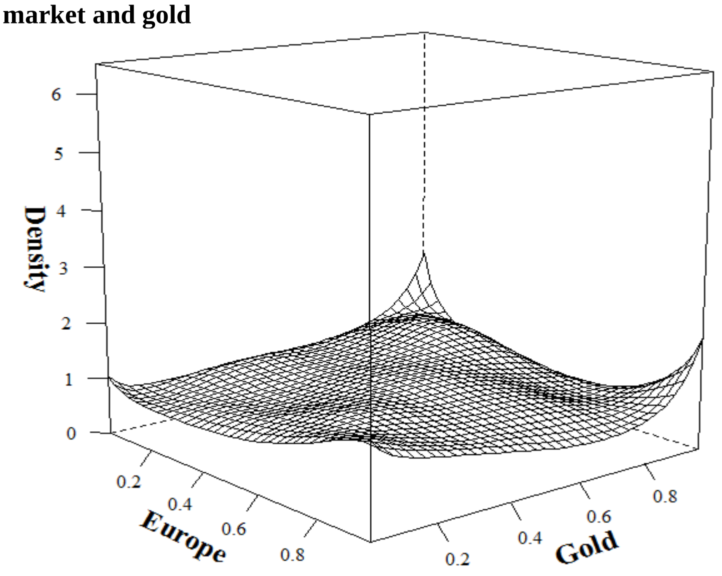


Figure 10E. Nonparametric copula for the US equity market and gold

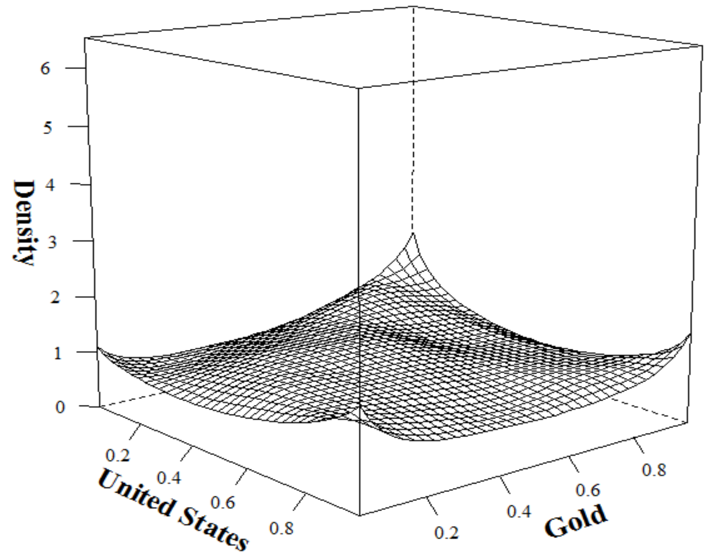
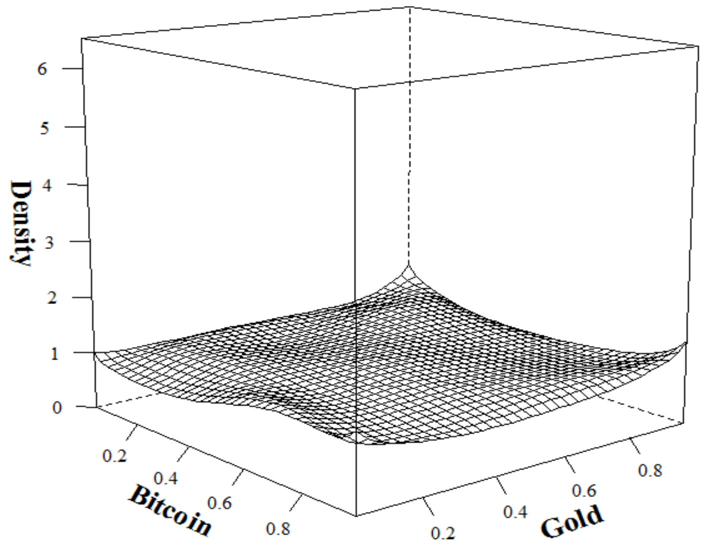


Figure 10F. Nonparametric copula for bitcoin and gold



Note: This figure represents the surface plots for the nonparametric kernel-type copula density estimator for the European and US equity markets, represented by the STOXX Europe 600 index and the S&P 500 index, with either bitcoin or gold. Figure 10A refers to equity markets, i.e., the European and US equity markets (Step 1). Figures 10B and 10C refer to equity markets and the bitcoin (Step 2), i.e., the European equity market and bitcoin and the US equity market and bitcoin, respectively. Figures 10D and 10E refer to equity markets and gold (Step 3), i.e., the European equity market and gold and the US equity market and gold, respectively. Figure 10F refers to bitcoin and gold (Step 4).